Professional Forecasters: How to Understand and Exploit Them Through a DSGE Model

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Professional Forecasters: How to Understand and Exploit Them Through a DSGE Model*

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Abstract

This paper derives a link between the forecasts of professional forecasters and a DSGE model. I show that the forecasts of a professional forecaster can be incorporated to the state space representation of the model by allowing the measurement error of the forecast and the structural shocks to be correlated. The parameters capturing this correlation are reduced form parameters that allow to address two issues i) How the forecasts of the professional forecaster can be exploited as a source of information for the estimation of the model and ii) How to characterize the deviations of the professional forecaster from an ideal complete information forecaster in terms of the shocks and the structure of the economy.

Keywords: Professional Forecasters, DSGE models.
JEL Code: C18, C51, C82, E17.

1 Introduction

The people in the economy are continuously forming and revising expectations, the majority is thinking about the probability of finding a job in the next month; or how much their salaries will rise; or the evolution of the interest rate of their debt. While some others, cause of the nature of their business, devote time and effort to form well informed expectations about macroeconomic aggregates: such as CPI inflation or GDP growth rate. People of the latter kind sometimes publish forecasts of economic variables–declare their expectations[1]–and there exists also surveys that collect these forecasts[2] such as the Federal Reserve and the European Central Bank surveys of professional forecasters.

The surveys have been used to characterize, from a merely statistical standpoint, the forecast accuracy and the forecast error of the professional forecasters (see Bowles, Friz, Genre, Kenny,

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[1]Although not every published forecast could be considered as some revealed expectations because of the different nature that may have the loss function of the forecaster. For the case of Professional forecasters see Ottaviani and Sorensen (2006), they show that the PF might have incentives to deviate from their best possible forecast.

[2]Some of the respondents of the surveys does not publish their forecast and their identification is not revealed when the results of the surveys are published. Therefore, they do not have the incentives discussed in Ottaviani and Sorensen (2006) to have forecasts different from their expectations.

Meyler, and Rautanen (2007) and Stark (2010)) and also as a source of information to construct atheoretical forecasting models (see Genre, Kenny, Meyler, and Timmermann (2010)). I depart from previous studies and derive a methodology that belongs to "Rational Expectations Econometrics" which Sargent (1989) refers to as:

> "Rational expectations econometrics" aims to interpret economic time series in terms of objects that are meaningful to economists, namely, parameters describing preferences, technologies, information sets, endowments, and equilibrium concepts or coordination mechanisms.

Using a Dynamic and Stochastic General Equilibrium models (DSGE) I address simultaneously two issues:

i) How the forecasts of the Professional forecasters (Henceforth PF) can be exploited as a source of information for the estimation of the model.

ii) Characterize the deviations of the PF from an ideal complete information forecaster in terms of the shocks and the structure of the economy.

For both issues I stand as an econometrician with a DSGE model for the economy and a set of observable variables that include the forecasts from the PF.

Previous articles that have addressed indirectly[3] the first topic are Giannoni and Boivin (2005) and González, Mahadeva, Rodríguez, and Rojas (2009). To our knowledge there is not in the literature a tentative answer to the second question.

Giannoni and Boivin (2005) show in a general form how to include a "rich" data set for the estimation of a DSGE model using the data as indicator variables of latent factors; quite a proper interpretation of the information contained in a forecast. Nevertheless, misses the particular details present in the case of PF that, as I show, if the purpose is i are relevant for the specification of the measurement equation[4] and to construct priors for the parameters.

González, Mahadeva, Rodríguez, and Rojas (2009) proposes a methodology to include data about the future, such as forecasts from other models, in a DSGE model for forecasting. The methodology don't incorporate the possible correlation of the measurement errors with the structural shocks of the model. I show here that in the case of PF this correlation emerges naturally, is informative and not negligible.

The strategy to solve the issues is based on two alternative specifications for the PF. The first one is a structural specific case and the other a reduced form general case. For the first type of PF I suppose that he differs with the econometrician only in the information set; for the general case the PF might have also a different model of the economy[5]. Regarding the first issue "i" I show for both specifications how to incorporate the forecasts of the PF as observable variables in the model and the implied log-likelihood function. It turns out that a specific structure of the measurement error must be specified with the main feature that the structural shocks of the model and the measurement error are correlated.

The first specification is as an extension to Sargent (1989), who shows how to obtain the likelihood function of the model for two different specifications of the statistical agency in charged to publish the data. I include also the PF, which could be thought as a statistical agency that also publishes forecasts. As the forecasts are not straightforward indicators of the state

[3]In a general framework and not referring specifically to PF.

[4]I refer here to the state space representation of the model.

[5]I refer to this case as the reduced form general case because I don't show explicitly the model of the economy that the PF has. Instead of this I show his reduced form forecast function.

of the economy the signal extraction problem is richer: reveals features of the PF relevant for estimation. On the other hand, the second specification is a general case of the previous extension that allow us to show the robustness of our findings.

I solve the second issue "*ii*" characterizing the corresponding measurement error term of the PF forecasts. The measurement errors are the deviations of the PF relative to an ideal forecaster and are found to follow a VAR(1) process. Furthermore, I show how the reduced form parameters contained in the VAR measure the magnitude and persistence in which each economic shock induce a difference between the PF and the ideal forecaster.

Previous articles such as Ireland (2004) have proposed to model the measurement errors with a VAR process, in what is known as the DSGE-VAR literature (see de Córdoba and Torres (2009)). In Ireland (2004) the shocks of the VAR process are exogenous innovations not related to the theoretical model. I depart from the previous studies because the shocks of the VAR process are found to be the structural shocks of the model and shocks related to the measurement errors of the PF information.

After describing a general setup with the model and filtering equations I present the likelihood function, the VAR process of the measurement errors and it´s reduced form parameters for each of the PF specifications. I provide concrete illustration of how this reduced form parameters capture the difference between the PF and the ideal complete information forecaster.

2 General Setup

Here I set some notation for the economic model, the filtering equations and the log-likelihood function. From this general setup the econometrician and the first specification of the PF are modeled.

There is an economic model with rational expectations whose equilibrium can be represented as a covariance stationary stochastic process. Specifically, the model equilibrium can be represented as

$$x_{t+1} = Tx_t + \varepsilon_t \tag{1}$$

where x_t is a $n \times 1$ vector of variables, the matrix T is a function of the parameters of the model and ε_t is a $n \times 1$ vector of structural shocks whose expected value and covariance matrices are characterized by:

$$\begin{aligned} E\{\varepsilon_t \varepsilon_s'\} &= \begin{cases} \mathbf{Q} & \text{for } t = s \\ \mathbf{0} & \text{for } t \neq s \end{cases} \\ E\{\varepsilon_t\} &= \mathbf{0} \end{aligned} \tag{2}$$

where $E\{.\}$ stands as the expectational operator. The economic model is completely represented by (1) and (2).

Related to those variables of the model there is a set of observable variables $\{y_0, y_1, \ldots, y_t, \ldots, y_\tau\}$, where y_t is a $k \times 1$ vector. These relationship is represented by:

$$y_t = Cx_t + \nu_t$$

where C is a $k \times n$ matrix that captures the linear projection of y_t over x_t. The $k \times 1$ vector ν_t is conformed by stochastic variables that model the movements of y_t not explained by Cx_t. ν_t is commonly known in these context as the vector of measurement errors as each element y_t is intended to "measure" some linear combination of x_t, and v_t stands as the deviation of y_t from that linear combination. The nature of the measurement errors v_t is determined by the following covariance matrices and expected value:

$$\begin{aligned} E\{\nu_t \nu_s'\} &= \begin{cases} \mathbf{R} & \text{for } t = s \\ \mathbf{0} & \text{for } t \neq s \end{cases} \\ E\{\nu_t\} &= \mathbf{0} \end{aligned}$$

Furthermore, in this general setup I assume that the structural shocks and the measurement errors are orthogonal at any point in time,

$$E\{\varepsilon_t \nu_s'\} = \mathbf{0} \text{ for all } t, s \tag{3}$$

Following a time-domain approach the state-space representation of the model is:

$$\begin{aligned} x_{t+1} &= Tx_t + \varepsilon_t \\ y_t &= Cx_t + \nu_t \end{aligned} \tag{4}$$

Where the first equation in (4) is the transition equation and the later corresponds to the measurement equation. This specification resembles to the "classical model of measurements initially collected by an agency" presented in Sargent (1989). Following Sargent (1989) the filtered variables can be obtained recursively by:

$$\begin{aligned} \hat{x}_t &= E(x_t | y_t, y_{t-1}, \ldots y_0, \hat{x}_0) \\ &= T\hat{x}_{t-1} + Ku_t \end{aligned} \tag{5}$$

where K is the gain matrix of the Kalman filter and u_t is the one-step ahead forecast error, or more formally

$$u_t = y_t - E\{y_t | y_{t-1}, y_{t-2}, \ldots\} \tag{6}$$

and I define

$$\begin{aligned} S &= E\left\{(\hat{x}_t - x_t)(\hat{x}_t - x_t)'\right\} \\ V &= E\{u_t u_t'\} \end{aligned}$$

then the Gaussian log-likelihood function for the sample $\{y_0, y_1, \ldots, y_t, \ldots, y_\tau\}$, conditioned on $\hat{x}_0$ is

$$L = -\tau \ \ln(2\pi) - 0.5 \ \ln|V| - 0.5 \sum_{t=0}^{\tau-1} u_t' V u_t$$

3 A Professional Forecaster With a Different Information Set

The information set of the PF and the econometrician might be different, one possible explanation for this is private information either of the PF or the econometrician. I'm interested, from the standpoint of the econometrician, to learn about the private information that may have the PF. In terms of the model, which shocks can identify the PF so the econometrician can use his forecasts as an information variable for estimation and forecasting. Also, the purpose is to explain the PF differences with an ideal complete information forecaster in terms of the shocks of the model that are poorly identified by the PF[6].

3.1 The Professional Forecaster

There is a PF who performs optimal forecasts[7] using the economic model mentioned and a data set $(y_0^f, y_1^f, \ldots, y_t^f, \ldots, y_\tau^f)$ where y_t^f is a $k \times 1$ vector of data related to the model variables by

$$
\begin{aligned}
y_t^f &= C^f x_t + \nu_t^f \\
E\left\{\nu_t^f \left(\nu_t^f\right)'\right\} &= \mathbf{R} \\
E\left\{\nu_t^f\right\} &= \mathbf{0}
\end{aligned}
$$

Then from (5) the optimal filtering of the PF is:

$$
\begin{aligned}
\hat{x}_t^f &= E(x_t | y_t^f, y_{t-1}^f, \ldots y_0^f, \hat{x}_0) \\
&= T\hat{x}_{t-1}^f + K^f u_t^f \\
S^f &= E\left\{\left(\hat{x}_t^f - x_t\right)\left(\hat{x}_t^f - x_t\right)'\right\}
\end{aligned} \tag{7}
$$

where K^f is the gain matrix of the PF. The one step ahead forecast is then

$$
\begin{aligned}
\hat{x}_{t+1|t}^f &= E(x_{t+1} | y_t^f, y_{t-1}^f, \ldots y_0^f, \hat{x}_0) \\
&= E(Tx_t + \varepsilon_t | y_t^f, y_{t-1}^f, \ldots y_0^f, \hat{x}_0) \\
&= T\, E(x_t | y_t^f, y_{t-1}^f, \ldots y_0^f, \hat{x}_0) + E(\varepsilon_t | y_t^f, y_{t-1}^f, \ldots y_0^f, \hat{x}_0) \\
&= T\hat{x}_t^f
\end{aligned} \tag{8}
$$

[6]Another possible reason that might generate different information sets between the PF and the econometrician is rational inattention. In the case of the PF, he might neglect part of the information that the econometrician have (or vice versa) not because is private but because it is costly to obtain or process it and the gains of including this information are not big enough. In this case the shocks poorly identified by the PF are possibly shocks less important to quantify for the PF. Mackowiak and Wiederholt (2009) shows how the firms might optimally decide not to identify an aggregate shock if the idiosyncratic shocks are more volatile.

[7]In the sense that minimizes the expected value of the squared forecast error.

where $E(\varepsilon_t|y_t^f, y_{t-1}^f, \ldots y_0^f, \hat{x}_0) = 0$ follows from (2) and (3).

The PF publishes the one-step ahead forecast of some variables each period. Define $\tilde{y}_t$ as the subset of $\hat{x}_{t+1|t}^f$ that is observable for the econometrician and published at time t, then

$$\tilde{y}_t = I_s \quad \hat{x}_{t+1|t}^f \tag{9}$$

where I_s is a selection matrix conformed by the rows of the identity matrix that correspond to a observable variable i.e the row j of the identity matrix is one of the rows of I_s if the entry j of $\hat{x}_{t+1|t}^f$ is published. Then, from (8) and (9), $\tilde{y}_t$ can be written in terms of the filtered values of the PF as

$$\tilde{y}_t = I_s T \hat{x}_t^f \tag{10}$$

3.2 Incorporating the forecasts from the PF

Suppose initially (for ease of exposition) that the econometrician only observes $\{\tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_t, \ldots, \tilde{y}_\tau\}$. From (10) and (7) follows a state-space representation with $\tilde{y}_t$ as the observable and $\hat{x}_t^f$ as the unobservable states. The transition and measurement equation of the representation are

$$\begin{aligned} \hat{x}_{t+1}^f &= T\hat{x}_t^f + K^f u_{t+1}^f \\ \tilde{y}_t &= I_s T \hat{x}_t^f \end{aligned} \tag{11}$$

The system (11) is in terms of the innovations u_t^f, and the unobservable states $\hat{x}_t^f$ that are the filtered values of the PF. On the other hand, using the law of motion of the variables in the model by (1), another possible state-space representation with the data $\tilde{y}_t$ as the observable and redefining the unobservable states as x_t can be written as follows

$$\begin{aligned} x_{t+1} &= Tx_t + \varepsilon_t \\ \tilde{y}_t &= I_s T x_t + v_t \end{aligned} \tag{12}$$

Now a measurement error $v_t = I_s T \left(\hat{x}_t^f - x_t\right)$ emerge. To understand the nature of this measurement error note that if the PF has complete information[8] then

$$\hat{x}_t^f = E\{x_t|x_t\} = x_t$$

and $v_t = 0$. So in this case the measurement error associated with the forecast of the PF reflects the difference between the forecast of the PF $I_s T \hat{x}_t^f$ and the forecast of a complete information forecaster $I_s T x_t$, this can be stated as

$$v_t = E\left\{x_{t+1}|y_t^f, y_{t-1}^f, \ldots y_0^f, \hat{x}_0\right\} - E\{x_{t+1}|x_t\}$$

thus v_t contains the signal extraction uncertainty of the PF.

[8]In the sense that knows perfectly the current state of the economy x_t but is uncertain about the shocks that can arrive $(\varepsilon_t, \varepsilon_{t+1}, \varepsilon_{t+2}, \ldots)$.

Defining $e_t = \varepsilon_{t-1}$ the contemporaneous form of the state-space representation is

$$\begin{aligned} x_t &= T x_{t-1} + e_t \\ \tilde{y}_t &= I_s T x_t + v_t \end{aligned}$$

In this case e_t and v_t are correlated, the covariance matrix is:

$$\begin{aligned} \Theta &= E\{v_t e_t'\} \\ \Theta &= I_s T \left(K^f C - I\right) \mathbf{Q} \end{aligned} \tag{13}$$

and the variance matrix of the measurement error is:

$$R = E\{v_t v_t'\} = I_s T S^f (I_s T)'$$

v_t is not the standard measurement error because it is autocorrelated. Formally,

$$\begin{aligned} E\{v_t v_{t-j}'\} &= E\left\{ I_s T \left(\hat{x}_t^f - x_t\right) \left(\hat{x}_{t-j}^f - x_{t-j}\right)' (I_s T)' \right\} \\ &= (I_s T) \left(\prod_{i=1}^{j} \left(I - K^f I_s T\right) T \right) S^f (I_s T)' \end{aligned} \tag{14}$$

The next proposition clarifies the nature of v_t. It resumes in a compact form the information presented in (13) and (14) and the relationship of v_t with ν_t^f.

Proposition 1. *The stochastic process $\{v_t\}_{t=1,\ldots,\infty}$ can be written as a vector autoregresive (VAR) process of the form:*

$$v_t = \Phi v_{t-1} + \Gamma e_t + \Omega \nu_t^f$$

where the matrices Φ, Γ and Ω correspond to:

$$\begin{aligned} \Phi &= I_s T \left(\left(I - K^f I_s\right) T\right) \left((I_s T)' I_s T\right)^{-1} (I_s T)' \\ \Gamma &= I_s T \left(K^f C - I\right) = \Theta Q^{-1} \\ \Omega &= I_s T K^f \end{aligned} \tag{15}$$

Proof. The measurement error v_t in equation (12) correspond to:

$$v_t = I_s T \left(\hat{x}_t^f - x_t\right)$$

replacing $\hat{x}_t^f$ using (11) and x_t using (12):

$$\begin{aligned} v_t &= I_s T \left(T \hat{x}_{t-1}^f + K^f u_t^f\right) - I_s T \left(T x_{t-1} + e_t\right) \\ &= I_s T T \left(\hat{x}_{t-1}^f - x_{t-1}\right) + I_s T K^f u_t^f - I_s T e_t \end{aligned}$$

replacing the one-step ahead forecast error u_t^f by it´s definition (see (6))

$$\begin{aligned} v_t &= I_s T T \left(\hat{x}_{t-1}^f - x_{t-1}\right) + I_s T K^f \left(y_t^f - C^f T \hat{x}_{t-1}^f\right) - I_s T e_t \\ &= I_s T T \left(\hat{x}_{t-1}^f - x_{t-1}\right) + I_s T K^f \left(C^f x_t + \nu_t^f - C^f T \hat{x}_{t-1}^f\right) - I_s T e_t \end{aligned}$$

using (12) to solve out for $x_t$and arranging terms

$$\begin{aligned} v_t &= I_s T T \left(\hat{x}_{t-1}^f - x_{t-1}\right) + I_s T K^f \left(C^f (T x_{t-1} + e_t) + \nu_t^f - C^f T \hat{x}_{t-1}^f\right) - I_s T e_t \\ &= I_s T (I - K^f C^f) T \left(\hat{x}_{t-1}^f - x_{t-1}\right) + I_s T K^f \nu_t^f + I_s T (K^f C^f - I) e_t \end{aligned}$$

using the definition of v_t,

$$\begin{aligned} v_t &= I_s T (I - K^f C^f) T \left((I_s T)' I_s T\right)^{-1} (I_s T)' I_s T \left(\hat{x}_{t-1}^f - x_{t-1}\right) + I_s T K^f \nu_t^f + I_s T (K^f C^f - I) e_t \\ &= I_s T (I - K^f C^f) T \left((I_s T)' I_s T\right)^{-1} (I_s T)' v_{t-1} + I_s T K^f \nu_t^f + I_s T (K^f C^f - I) e_t \end{aligned}$$

□

The matrices in (15) fully characterize the deviations of the PF from the ideal forecaster; their entries are reduced form parameters that are functions of the parameters of the model and the PF parameters (specifically the PF gain matrix). The matrix Γ measures the effect that has each structural shock in v_t , as v_t arises because of the lack of information of the PF, the entries in Γ reflect the uncertainty of the PF over the corresponding shock, weighted by the importance of it on the variable to forecast. On the other hand, Φ measures how the deviations v_t affect v_{t+1}, or in other terms, it captures the persistence structure of the deviations of the PF from the ideal forecaster. (15) show that the persistence depends on the structure of the economy T and the learning process of the PF K^f. If the economy has low persistence and the PF learns fast, the persistence of v_t will tend to zero. Finally Ω captures how the measurement error of the data used by the PF is translated to v_t[9].

Proposition 1 can explain the five characteristics of the PF that Andrade and Le Bihan (2010) found in the European Central Bank survey: "[PF ...] (i) have predictable forecast errors; (ii) disagree; (iii) fail to systematically update their forecasts in the wake of new information; (iv) disagree even when updating; and (v) differ in their frequency of updating and forecast performances" (Andrade and Le Bihan (2010)).

The first characteristic follows from the autoregresive component of v_t[10]. Second, If the PF differ in the information sets, ii, iv and v are explained because the matrices in 15 are different, it implies they "disagree" ($\hat{x}_t^f$ is different for each of them) and are different "updating" or learning (Φ and Γ differ between forecasters). Finally, iii is explained by the correlation of v_t and e_t, if the "new" information is about the shocks that the PF poorly identifies it will not be completely incorporated in the forecasts.

[9]For practical purposes, as is generally not known which data used the PF and consequently the size and elements of ν_t^f are not known, 15 can be writen in terms of the reduced form vector $\psi_t = \Omega \nu_t^f$; which covariance matrix would reflect the data uncertainty of each of the forecasts. Therefore, v_t can be written as $v_t = \Phi v_{t-1} + \Gamma e_t + \psi_t$.

[10]v_t is not the actual forecast error, although is a component of it. The shocks that arrive to the economy in the forecast horizon conform the other component; by the definition of the shocks, not predictable.

An Extended Data Base

Now I will extend the initial formulation to allow for a more general set of information for the econometrician. Collecting our results

$$
\begin{aligned}
x_t &= T x_{t-1} + e_t \\
v_t &= \Phi v_{t-1} + \Gamma e_t + \Omega \nu_t^f \\
\tilde{y}_t &= I_s T x_t + v_t
\end{aligned}
$$

y_t is the data released at time t which is composed by:

$$
y_t = \begin{pmatrix} d_t \\ \tilde{y}_t \end{pmatrix} \tag{16}
$$

where d_t is data related to the variables of the model and $\tilde{y}_t$ is the vector of the one-step ahead forecasts of the PF. Now the measurement equation is

$$
y_t = \begin{pmatrix} N \\ I_s T \end{pmatrix} x_t + \begin{pmatrix} \mu_t \\ v_t \end{pmatrix}
$$

where N is a matrix that captures the relation between the variables in the model and the data contained in d_t. μ_t is a vector of the measurement errors associated with d_t. Then, a complete formulation of the state space representation is

$$
\begin{aligned}
x_t &= T x_{t-1} + e_t \\
v_t &= \Phi v_{t-1} + \Gamma e_t + \Omega \nu_t^f \\
y_t &= \begin{pmatrix} N \\ T_{(i \in B)} \end{pmatrix} x_t + \begin{pmatrix} \mu_t \\ v_t \end{pmatrix} \\
E\{e_t e_t'\} &= Q \qquad E\{e_t\} = \mathbf{0} \\
E\{\mu_t \mu_t'\} &= H \qquad E\{\mu_t\} = \mathbf{0} \\
E\left\{\nu_t^f \left(\nu_t^f\right)'\right\} &= \mathbf{R} \qquad E\left\{\nu_t^f\right\} = \mathbf{0} \\
E\{\mu_t e_s'\} &= \mathbf{0} \text{ for all } t, s \\
E\left\{\nu_t^f e_s'\right\} &= \mathbf{0} \text{ for all } t, s
\end{aligned} \tag{17}
$$

3.3 The Log-Likelihood function neglecting Φ

The more recent innovations might be the main drivers of the measurement errors of the PF forecasts (i.e the discrepancy between the PF and the ideal complete information forecaster). If this is the case v_t will be mainly explained by Γe_t and the term Φv_{t-1} could be neglected, then the state space representation can be restated as

$$
\begin{aligned}
x_t &= Tx_{t-1} + \begin{pmatrix} \mathbf{0} & \mathbf{0} & I \end{pmatrix} \begin{pmatrix} \mu_t \\ \nu_t^f \\ e_t \end{pmatrix} \\
y_t &= \begin{pmatrix} N \\ I_s T \end{pmatrix} x_t + \begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega & \Gamma \end{pmatrix} \begin{pmatrix} \mu_t \\ \nu_t^f \\ e_t \end{pmatrix} \\
E\left\{ \begin{pmatrix} \mu_t \\ \nu_t^f \\ e_t \end{pmatrix} \begin{pmatrix} \mu_t' & \left(\nu_t^f\right)' & e_t' \end{pmatrix} \right\} &= \begin{pmatrix} H & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & Q \end{pmatrix} = \begin{pmatrix} h\,h' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r\,r' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & q\,q' \end{pmatrix} \\
E\left\{ \begin{pmatrix} \mu_t \\ \nu_t^f \\ e_t \end{pmatrix} \right\} &= \mathbf{0}
\end{aligned}
$$

where h, r and q are obtained from the Cholesky decomposition of H, $\mathbf{R}$ and Q respectively. In this specification is evident the correlation between the measurement errors and the structural shocks. Furthermore, the state space representation in terms of the orthogonal shocks (ζ_t) is

$$
\begin{aligned}
x_t &= Tx_{t-1} + \begin{pmatrix} \mathbf{0} & r & q \end{pmatrix} \zeta_t \\
y_t &= \begin{pmatrix} N \\ I_s T \end{pmatrix} x_t + \begin{pmatrix} h & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega r & \Gamma q \end{pmatrix} \zeta_t \\
E\left\{\zeta_t \zeta_t'\right\} &= I \quad E\zeta_t = \mathbf{0}
\end{aligned}
$$

or in a compact form

$$
\begin{aligned}
x_t &= \mathbf{T} x_{t-1} + \mathbf{H}\zeta_t \\
y_t &= \mathbf{Z} x_t + \mathbf{G}\zeta_t \\
E\left\{\zeta_t \zeta_t'\right\} &= I \quad E\zeta_t = \mathbf{0}
\end{aligned}
$$

This particular state-space form and the respective Kalman filter and smoother recursions can be found in Koopman and Harvey (2003). From there the filtered variables can be obtained by:

$$
\begin{aligned}
\hat{x}_t &= E(x_t | y_t, y_{t-1}, \ldots y_0, \hat{x}_0) \\
&= T\hat{x}_t + K a_{t+1} \\
E\left\{ (\hat{x}_t - x_t)(\hat{x}_t - x_t)' \right\} &= \mathbf{S}
\end{aligned}
$$

where K is the gain matrix of the Kalman filter and a_t is the one-step ahead forecast error, or more formally

$$
\begin{aligned}
K &= \left(\mathbf{TS}(\mathbf{ZT})' + \mathbf{H}(\mathbf{G} + \mathbf{ZH})' \right) \mathbf{V}^{-1} \\
a_t &= y_t - E\left\{ y_t | y_{t-1}, y_{t-2}, \ldots \right\} \\
E\left\{ a_t a_t' \right\} &= \mathbf{V} = \left(\mathbf{ZTS}(\mathbf{ZT})' + (\mathbf{G} + \mathbf{ZH})(\mathbf{G} + \mathbf{ZH})' \right)
\end{aligned}
$$

Then the Gaussian log-likelihood function for the sample $\{y_0, y_1, \ldots, y_t, \ldots, y_\tau\}$, conditioned on $\hat{x}_0$ is

$$L = -\tau\ \ln(2\pi) - 0.5\ \ln|\mathbf{V}| - 0.5\sum_{t=0}^{T-1} a_t'\mathbf{V}a_t$$

With the log-likelihood function the reduced form parameters contained in Γ and Ω (and the deep parameters too) can be estimated by maximum likelihood or with Bayesian techniques considering the possible characteristics of the gain matrix of the PF to construct the priors. The reduced form approach is very useful in this scenario for the parameters in Γ and Ω because typically K^f is not observable although there might have some prior knowledge about it.

3.4 The Log-Likelihood function, general form

To obtain the Likelihood function of (17) allowing the matrix Φ to be different from a null matrix I restate the state space representation (17) as follows

$$\begin{aligned}
\begin{pmatrix} x_t \\ v_t \end{pmatrix} = s_t &= \begin{pmatrix} T & \mathbf{0} \\ \mathbf{0} & \Phi \end{pmatrix} s_{t-1} + \begin{pmatrix} I & \mathbf{0} \\ \Gamma & \Omega \end{pmatrix} \begin{pmatrix} e_t \\ \nu_t^f \end{pmatrix} \\
y_t &= \begin{pmatrix} N & \mathbf{0} \\ I_s T & I \end{pmatrix} s_t + \begin{pmatrix} I \\ \mathbf{0} \end{pmatrix} \mu_t \\
E\left\{ \begin{pmatrix} e_t \\ \nu_t^f \end{pmatrix} \begin{pmatrix} e_t' & \left(\nu_t^f\right)' \end{pmatrix} \right\} &= \begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{pmatrix} \\
E\left\{ \begin{pmatrix} e_t \\ \nu_t^f \end{pmatrix} \right\} &= \mathbf{0} \\
E\{\mu_t \mu_t'\} &= H \quad E\{\mu_t\} = \mathbf{0} \\
E\left\{ \mu_t \begin{pmatrix} e_s' & \left(\nu_t^f\right)' \end{pmatrix} \right\} &= \mathbf{0} \quad \forall t, s
\end{aligned} \tag{18}$$

or in a compact form

$$\begin{aligned}
s_t &= \mathbf{T}s_{t-1} + \mathbf{L}\omega_t \\
y_t &= \mathbf{Z}s_t + \mathbf{B}\mu_t \\
E\{\omega_t \omega_t'\} &= \begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{pmatrix} \quad E\{\omega_t\} = \mathbf{0} \\
E\{\mu_t \mu_t'\} &= H \quad E\{\mu_t\} = \mathbf{0} \\
E\{\mu_t e_s\} &= \mathbf{0} \text{ for all } t, s
\end{aligned} \tag{19}$$

with this specification the filtered variables can be obtained by:

$$\begin{aligned}
\hat{s}_t &= E(s_t | y_t, y_{t-1}, \ldots y_0, \hat{x}_0) \\
&= T\hat{s}_t + K a_{t+1} \\
\mathbf{S} &= E\left\{ (\hat{s}_t - s_t)(\hat{s}_t - s_t)' \right\}
\end{aligned}$$

where K is the gain matrix of the Kalman filter and a_t is the one-step ahead forecast error, or more formally

$$\begin{aligned} a_t &= y_t - E\{y_t|y_{t-1}, y_{t-2}, \ldots\} \\ \mathbf{V} &= E\{a_t a_t'\} \end{aligned}$$

Then the Gaussian log-likelihood function for the sample $\{y_0, y_1, \ldots, y_t, \ldots, y_\tau\}$, conditioned on $\hat{s}_0$ is

$$L = -\tau \ \ln(2\pi) - 0.5 \ \ln|\mathbf{V}| - 0.5\sum_{t=0}^{T-1} a_t'\mathbf{V}a_t$$

With the log-likelihood function the reduced form parameters contained in Γ and Φ can be estimated by maximum likelihood or with Bayesian techniques. Again, the explicit form of Γ, Ω and Φ is an important feature for setting the priors for the estimation. Incorporating Φ allow us to think about the speed of learning of the PF.

4 A PF with a different forecasting model

Until this point the PF constructs his optimal forecasts using the same economic model as the econometrician, perhaps a strong assumption. This section extends the derivation for the case in which the forecast function can be approximated by a linear function of the data considered by the PF:

$$x^f_{t+1|t} = F\, y^f_t$$

where F is a matrix that contains the set of weights that the PF assigns to each piece of data contained on y^f_t. This specification does not necessarily impose the restriction that the PF only considers the latest released data because y^f_t might include lags of some variables. This data is related to the variables of the model by[11]:

$$y^f_t = C^f x_t + \nu^f_t$$

where ν_t is the vector of measurement errors. Then

$$x^f_{t+1|t} = F\, C^f x_t + F\nu^f_t$$

4.1 Incorporating the forecasts from the PF

Starting with the case where the only observable variables are the one-step ahead forecasts of some variables,

$$\begin{aligned} \tilde{y}_t &= x^f_{t+1|t} \\ &= F\, C^f x_t + F\nu^f_t \end{aligned} \tag{20}$$

[11]Again, here I could extend vector x_t to include lags of some relevant model variables in case some of the data is lagged.

with $E\left\{\nu_t^f\left(\nu_t^f\right)'\right\}=\bar{H}$. (20) can be written in terms of the expectations of the agents in the model in the form:

$$
\begin{aligned}
\tilde{y}_t &= I_s T x_t + m_t + v_t^f \\
m_t &= (F\,C^f - I_s T)x_t \\
v_t^f &= F\nu_t^f \\
\bar{\mathbf{H}} &= E\left\{v_t^f\left(v_t^f\right)'\right\} = F\bar{H}F'
\end{aligned}
$$

$m_t + v_t^f$ can be interpreted as a model mismatch error. The model mismatch error characterizes the difference between the forecast from the PF and the complete information forecast at time t, it can be written as:

$$
m_t + \nu_t^f = I_s x_{t+1|t}^f - E\left\{I_s x_{t+1}|x_t, x_{t-1}, ...\right\}
$$

The model mismatch term emerges in two cases i) if the forecaster has a different model of the economy or ii) If the forecaster has no complete information. The latter case has been covered in the third section, this section extends the formulation to incorporate also the first case. The shortcoming of the approach is that our results rely on terms such as F which are not "structural" strictly speaking. Nevertheless, it allows us to show that the reduced form parameters obtained in the previous section also emerge in this more general setup.

Analogous to Proposition 1 the stochastic process $\{m_t\}_{t=1,\ldots,\infty}$ can be represented in the form

$$
\begin{aligned}
m_t &= \bar{\Phi} m_{t-1} + \bar{\Gamma} e_t \\
\bar{\Phi} &= (F\,C - I_s T)\,T\left[(F\,C - I_s T)'\,(F\,C - I_s T)\right]^{-1}(F\,C - I_s T)' \\
\bar{\Gamma} &= (F\,C - I_s T)
\end{aligned} \tag{21}
$$

(21) shows that the magnitude and sign of the model mismatch term depends on the type of shocks present in the economy. The PF, depending on the shocks, might have his forecast near or far from the optimal complete information forecast.

Collecting our results the state-space representation of the model is:

$$
\begin{aligned}
x_t &= T x_{t-1} + e_t \\
m_t &= \bar{\Phi} m_{t-1} + \bar{\Gamma} e_t \\
\tilde{y}_t &= I_s T x_t + m_t + v_t \\
E\{e_t e_t'\} &= Q \qquad E\{e_t\} = \mathbf{0} \\
E\{v_t v_t'\} &= \mathbf{H} \qquad E\{v_t\} = \mathbf{0}
\end{aligned}
$$

and with a more general vector of observable variables y_t defined in (16)

$$\begin{aligned}
x_t &= Tx_{t-1} + e_t \\
m_t &= \bar{\Phi} m_{t-1} + \bar{\Gamma} e_t \\
y_t &= \begin{pmatrix} N \\ I_s T \end{pmatrix} x_t + \begin{pmatrix} \mathbf{0} \\ I \end{pmatrix} m_t + \begin{pmatrix} \mu_t \\ v_t \end{pmatrix} \\
E\{e_t e_t'\} &= Q \qquad E\{e_t\} = \mathbf{0} \\
E\{v_t v_t'\} &= \bar{\mathbf{H}} \qquad E\{v_t\} = \mathbf{0} \\
E\{\mu_t \mu_t'\} &= H \qquad E\{\mu_t\} = \mathbf{0}
\end{aligned} \tag{22}$$

Obtaining the likelihood function of (22) is analogous to the steps shown for (17). Again, the Likelihood function depends on the reduced form parameters contained in $\bar{\Phi}$ and $\bar{\Gamma}$. So basically, to incorporate an outsider forecasts as observables for signal extraction, there should be specified a measurement error that is the sum of a standard measurement error term v_t and an autocorrelated and correlated with the structural shocks term m_t.

5 Conclusions

In "Rational Expectations Econometrics" the forecasts of professional forecasters can be used as sources of information for model estimation and to characterize the professional forecaster underlying signal extraction mechanism. The main feature that must be incorporated is the correlation of the measurement errors and the shocks of the model. Our proposal for the stochastic process of the measurement errors is a VAR(1); where the innovations are the shocks of the model. The VAR(1) characterize the PF coherently with the evidence found in the literature and provide a benchmark for the evaluation of the forecasters: a complete information forecaster.

The reduced form of the VAR allow to obtain the Log-Likelihood function of the DSGE model incorporating the PF forecasts as observables and also the reduced form parameters characterize the shocks of the economy that the professional forecasters miss (or don't learn about them). The explicit dependence shown of the reduced form parameters of the gain matrix of the PF and the structure of the economy is relevant information to construct priors for this parameters.

References

Andrade, P., and H. Le Bihan (2010): "Inattentive professional forecasters," Working papers 307, Banque de France.

Bowles, C., R. Friz, V. Genre, G. Kenny, A. Meyler, and T. Rautanen (2007): "The ECB survey of professional forecasters (SPF). A review after eight years experience," Occasional Paper Series 59, European Central Bank.

de Córdoba, G. F., and J. L. Torres (2009): "Forecasting the Spanish economy with an Augmented VAR-DSGE model," Working Papers 2009-1, Universidad de Málaga, Department of Economic Theory, Málaga Economic Theory Research Center.

GENRE, V., G. KENNY, A. MEYLER, AND A. TIMMERMANN (2010): "Combining the forecasts in the ECB survey of professional forecasters: can anything beat the simple average?," Working Paper Series 1277, European Central Bank.

GIANNONI, M. P., AND J. BOIVIN (2005): "DSGE Models in a Data-Rich Environment," *Computing in Economics and Finance*, (431).

GONZÁLEZ, A., L. MAHADEVA, D. RODRÍGUEZ, AND L. E. ROJAS (2009): "Monetary Policy Forecasting In A Dsge Model With Data That Is Uncertain, Unbalanced And About The Future," BORRADORES DE ECONOMIA 005480, BANCO DE LA REPÚBLICA.

IRELAND, P. N. (2004): "A method for taking models to the data," *Journal of Economic Dynamics and Control*, 28(6), 1205–1226.

KOOPMAN, S. J., AND A. HARVEY (2003): "Computing observation weights for signal extraction and filtering," *Journal of Economic Dynamics and Control*, 27(7), 1317–1333.

MACKOWIAK, B., AND M. WIEDERHOLT (2009): "Optimal Sticky Prices under Rational Inattention," *American Economic Review*, 99(3), 769–803.

OTTAVIANI, M., AND P. N. SORENSEN (2006): "The strategy of professional forecasting," *Journal of Financial Economics*, 81(2), 441–466.

SARGENT, T. J. (1989): "Two Models of Measurements and the Investment Accelerator," *Journal of Political Economy*, 97(2), 251–87.

STARK, T. (2010): "Realistic evaluation of real-time forecasts in the Survey of Professional Forecasters," *Research Rap Special Report*, (May).