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Investment horizon dependent CAPM: Adjusting beta for long-term dependence [‡]

Carlos León[1], Karen Leiton[2], Alejandro Reveiz[3]

Abstract

Financial basics and intuition stresses the importance of investment horizon for risk management and asset allocation. However, the beta parameter of the Capital Asset Pricing Model (CAPM) is invariant to the holding period. Such contradiction is due to the assumption of long-term independence of financial returns; an assumption that has been proven erroneous.

Following concerns regarding the impact of the long-term dependence assumption on risk (Holton, 1992), this paper quantifies and fixes the CAPM's bias resulting from this abiding –but flawed- assumption. The proposed procedure is based on Greene and Fielitz (1980) seminal work on the application of fractional Brownian motion to CAPM, and on a revised technique for estimating time-series' fractal dimension with the Hurst exponent (León and Vivas, 2010; León and Reveiz, 2011a).

Using a set of 85 stocks from the S&P100, this paper finds that relaxing the long-term independence assumption results in significantly different estimations of beta. According to three tests herein implemented with a 99% confidence level, more than 60% of the stocks exhibit significantly different beta parameters. Hence, expected returns are biased; on average, the bias is about $\pm$60bps for a contemporary one-year investment horizon.

Thus, as emphasized by Holton (1992), risk is a two-dimensional quantity, with holding period almost as important as asset class. The procedure herein proposed is valuable since it parsimoniously achieves an investment horizon dependent CAPM.

Key words: CAPM, Hurst exponent, long-term dependence, fractional Brownian motion, asset allocation, investment horizon.

JEL classification: G12, G14, G32, G20, C14

[‡] As usual, the opinions and statements are the sole responsibility of the authors.

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1. Introduction

The Capital Asset Pricing Model (CAPM) is one of the milestones of modern financial theory, and was the first model introducing a specific-asset risk concept in a general equilibrium model for asset pricing (Litterman, 2003). This model has endured for decades, and is not only a basic topic in traditional corporate finance courses, but continues as market practitioners' standard for several purposes.[4]

Academics and practitioners are aware of the most evident and well-known shortcomings related to the model's assumptions, such as the presence of non-normal returns, transaction costs, non-homogeneous market expectations, among others. However, one of the assumptions overlooked by academics and practitioners alike is that CAPM was created under the strong assumption of long-term independence of financial returns. This assumption results in the temporal consistency of volatility[5], and allows CAPM formulae not to consider the investment horizon (i.e. investor's holding period) as a relevant input when estimating expected returns; this is why, despite being highly counter-intuitive, an asset manager or a financial officer using CAPM is never concerned about what his client's or his own investment horizon is.

Because volatility may not be consistent over time, several authors (Holton, 1992; Peters, 1989; León and Reveiz, 2011a) have highlighted the perils resulting from overlooking the relevance of the investment horizon in the absence of time series' long-term independence, especially for asset pricing, asset allocation and risk management. Such variability of risk along the investment horizon contradicts traditional indifference with respect to investor's holding period, and urges for a methodological adjustment that introduces the investment horizon as a significant input for CAPM.

Holton (1992), León and Vivas (2010) and León and Reveiz (2011a) show that the main empirical consequence of long-term dependence is that it invalidates the main assumptions of Brownian motion, with major theoretical and empirical implications. For financial theory, according to Peters (1989) and Lo (1991), the existence of long-term memory rejects the Efficient Market Hypothesis (EMH); for financial practice, the square-root-of-time-rule to scale volatility to different maturities turns invalid. In both cases, outcomes in standard portfolio optimization and risk management models can change when financial time-series' long-term memory is considered.

[4] For instance, Mandelbrot and Hudson (2004) report that a 1999 survey conducted by two students from Duke University to 392 CFO listed in the U.S Fortune 500 yielded that 73.5% of the surveyed estimated capital cost using standard CAPM; in Europe, this percentage was 77% in 2001.
[5] As pointed out by Holton (1992), based on random walk theory, investment professionals assume temporal consistency of volatility, where, for instance, daily volatilities contain precisely the same information as five-year volatilities.

Early evidence of long-term dependence or memory in financial time series was reported by Mandelbrot (1972); he designed the Hurst exponent, a parameter that assesses the degree of long-term dependence.[6] Afterwards, other authors confirmed Mandelbrot's findings for developed markets (Peters, 1989 and 1992; León and Reveiz, 2011a; Ambrose *et al.*, 1993; Bilel and Nadhem, 2009) and for emerging markets (León and Vivas, 2010; Cajueiro and Tabak, 2008; Leiton, 2011). Consequently, temporal consistency of volatility assumption and investment horizon neutrality should not be taken for granted.

Hence, taking into account extensive evidence of the presence of long-term dependence and the relevance of CAPM for financial industry, this paper combines seminal work on the application of long-term memory processes to CAPM (Greene and Fielitz, 1979 and 1980), and a revised technique for estimating the Hurst exponent (León and Vivas, 2010; León and Reveiz 2011a), in order to propose a parsimonious method for adjusting traditional CAPM model for long-term dependence. The main outcomes of this method are investment-horizon-adjusted estimations of systemic risk and expected returns, along with the assessment of the bias resulting from the long-term independence assumption.

Using a set of 85 stocks from the S&P100, this paper finds that relaxing the long-term independence assumption results in significantly different estimations of beta. According to three significance tests herein implemented with a 99% confidence level, more than 60% of the stocks exhibit significantly different beta parameters. Hence, expected returns are biased; on average, the bias is about ±60bps for a contemporary one-year investment horizon.

Thus, as emphasized by Holton (1992), risk is a two-dimensional quantity, with holding period almost as important as asset class. The procedure herein proposed is valuable since it parsimoniously achieves an investment horizon dependent CAPM, which is particularly relevant for long-term investors and asset managers because it allows for designing allocations and making decisions that correspond to holding periods.

This paper consists of five sections; this introduction is the first one. The second section describes and develops classic rescaled range analysis (*R/S*) methodology for detecting and assessing the presence of long-term serial dependence of returns, along with the adjusted version proposed by León and Vivas (2010) and León and Reveiz (2011a). The third section describes and develops the adjustment proposed by Greene and Fielitz (1979 and 1980) for incorporating long-term dependence to the CAPM model. The fourth presents numerical evidence of the bias resulting from overlooking long-term dependence of financial time series. Finally, the last section highlights and discusses some relevant remarks.

[6] However, the Hurst exponent is not the only technique available to verify the existence of long-term dependence. As acknowledged by Malavergne and Sornette (2006), slow convergence in distribution of financial time-series to a Gaussian law –even for low frequency returns- may be a sign of significant time-dependencies between asset returns.

2. Assessing long-term dependence in financial time series[7]

Long-term dependence detection and assessment for time-series began with Hydrology (Mandelbrot and Wallis, 1969a), when the British physicist H.E. Hurst (1880–1978) was appointed to design a water reservoir on the Nile River. The first problem Hurst had to deal with was to determine the optimal storage capacity of the reservoir; that is, restricted to a budgetary constraint, design a dam high enough to allow for fluctuations in the water supply whilst maintaining a constant flow of water below the dam.

Deciding on the optimal storage capacity depended on the inflows of the river, which were customarily assumed to be random and independent by hydraulic engineers at that time. However, when checking the Nile's historical records (622 B.C. - 1469 B.C.) Hurst discovered that flows couldn't be described as random and independent: data exhibited persistence, where years of high (low) discharges were followed by years of high discharges (low), thus describing cycles but without an obvious periodicity.

Hurst concluded that (i) evidence contradicted the long-established independence assumption and (ii) that the absence of significant autocorrelation proved standard econometrics tests to be ineffective (Peters, 1994). Thus, since absence of independence vindicated caring about the size and sequence of flows, Hurst developed a methodology capable of capturing and assessing the type of dependence he had documented.

Hurst's methodological development was based on Einstein's (1905) work about particles' movement, which Scottish botanist Robert Brown (1828 and 1829) already depicted as inexplicable, irregular and independent. Einstein originally formulated that the distance or average displacement *(R)* covered by a particle suspended in a fluid per unit of time *(n)* followed $R = n^{0.5}$; this is analogous to financial industry's standard square-root-of-time rule for escalating volatility.[8]

Unlike Brown and Einstein, Hurst's primary objective was a broad formulae, capable of describing the distance covered by any random variable with respect to time. Hurst found his observations of several time-series were well represented by $R \sim c \times n^{H}$, where H corresponds to the way that distance *(R)* behaves with respect to time.

Hurst defined that the metric for the distance covered per unit of time or sample *(n)* would be given by the range R_n [F1], where $x_1, x_2, x_3 \ldots x_n$ correspond to the change of the random variable within the sample, and $\bar{x}_n$ is the average of these changes. Range R_n is

[7] This section is extracted from León and Reveiz (2011a).

[8] The square-root-of-time-rule consists of multiplying the standard deviation calculated from high-frequency time-series (e.g. daily) by the square-root of *n*, where *n* is the number of units which compose the low-frequency time-series (e.g. yearly). According to Sornette (2003), this is equivalent to saying that the typical amplitude of returns is proportional to the square-root-of-the-time scale, which he describes as the most important prediction of the Brownian motion model.

standardized by the standard deviation of the sample for that period *(S_n)*, which results in the rescaled range for the *n* sample *$(R/S)_n$* [F1].

$$(R/S)_n = {}^{R_n}/_{S_n} = \frac{\left[\max\limits_{1\le k\le n}\left(\sum_{j=1}^{k}(x_j-\bar{x}_n)\right)-\min\limits_{1\le k\le n}\left(\sum_{j=1}^{k}(x_j-\bar{x}_n)\right)\right]}{S_n} \qquad \text{[F1]}$$

Hurst found out that the behavior of this rescaled range [F1] adequately fitted the dynamic of numerous time-series from natural phenomena, where the fit could be represented as follows [F2]:

$$(R/S)_n \sim c \times n^H \qquad \text{[F2]}$$

Paraphrasing Peters (1992), Hurst's novel methodology measures the cumulative deviation from the mean for various periods of time and examines how the range of this deviation scales over time. $\widehat{H}$, the estimated exponent that measures the way distance *(R)* behaves with respect to time, takes values within the 0 and 1 interval *($0 < \widehat{H} \le 1$)*, where *$\widehat{H}=0.5$* corresponds to Einstein's and Brown's independency case.

Mandelbrot and Wallis (1969a and 1969b) proposed to plot Hurst's function [F2] for several sample sizes *(n)* in a double logarithmic scale, which served to obtain $\widehat{H}$ through a least squares regression. $\widehat{H}$ would be the slope of the estimated equation [F3]; this procedure is known as the rescaled range analysis (*R/S*).[9]

$$Log(R/S)_n = Log(c) + HLog(n) \qquad \text{[F3]}$$

According to Mandelbrot (1965) the application of *R/S* to random series with stationary and independent increases, such as those characterized by Brown (1828 and 1829) and Einstein (1905), results in *$\widehat{H}=0.5$*, even if the distribution of the stochastic process isn't Gaussian, in which case $\widehat{H}$ asymptotically converges to 0.5 *($\widehat{H} \approx 0.5$)*.

As stated by Sun *et al.* (2007), in the *$\widehat{H}=0.5$* and *$\widehat{H} \approx 0.5$* cases the process has no memory (i.e. is independent), hence next period's expected result has the same probability of being lower or higher than the current result. Applied to financial time-series this is akin to assuming that the process followed by assets' returns is similar to coin tossing, where the probability of heads (rise in the price) or tails (fall in the price) is the same *(½)*, and is independent of every other toss; this is precisely the theoretical base of the CAPM, the Arbitrage Pricing Theory (APT), the Black & Scholes model and the Modern Portfolio Theory (MPT).

When $\widehat{H}$ takes values between 0.5 and 1 *($0.5 < \widehat{H} \le 1$)* evidence suggests a persistent behavior, therefore one should expect the result in the next period to be similar to the current one (Sun *et al.*, 2007). According to Menkens (2007) this means that increments are positively correlated: if an increment is positive, succeeding increments are most likely

[9] León and Reveiz (2011a) include a step-by-step explanation on how to implement the *R/S* algorithm.

to be positive than negative. In other words, each event has influence on future events; therefore there is dependence or memory in the process.

As $\widehat{H}$ becomes closer to one (1) the range of possible future values of the variable will be wider than the range of purely random variables. Peters (1996) argues that the presence of persistency is a signal that today's behavior doesn't influence the near future only, but the distant future as well.

On the other hand, when $\widehat{H}$ takes values below 0.5 $(0 \leq \widehat{H} < 0.5)$ there is a signal that suggests an antipersistent behavior of the variable. This means, as suggested by Sun *et al.* (2007), that a positive (negative) return is more likely followed by negative (positive) ones; hence, as stated by Mandelbrot and Wallis (1969a), this behavior causes the values of the variable to tend to compensate with each other, avoiding time-series' overshooting. Applied to financial markets series, Menkens (2007) affirms that this kind of continuously compensating behavior would suggest a constant over-correction of the market, one that would drive it to a permanent adjustment process. Similarly, Peters (1996) links this behavior to the well-known "mean-reversion" process.

Hurst's methodology and results [10] were gathered, corrected and reinterpreted by Mandelbrot (1972) and Mandelbrot and Wallis (1969a and 1969b). Based on random simulation models they verified that (i) Hurst's conclusions were correct, but calculations were imprecise; (ii) their corrected version of *R/S* is robust to detect and measure dependence, even in presence of significant excess skewness or kurtosis[11]; (iii) their corrected version of *R/S* is asymptotically robust to short-term dependency (e.g. autoregressive and moving average processes); (iv) asymptotically $\widehat{H}=0.5$ for independent processes, even in absence of Gaussian processes; and (v) in contrast to other methodologies, *R/S* can detect non periodic cycles.

Some explanations for financial assets' return persistence are found in human behavior, since the latter contradicts rationality assumption in several ways, for example: (i) investors' choices are not independent, and they are characterized by non-linear and imitative behavior (LeBaron and Yamamoto, 2007; Sornette, 2003); (ii) investors resist changing their perception until a new credible trend is established (Singh and Dey, 2002; Peters, 1996), and (iii) investors don't react to new information in a continuous manner, but rather in a discrete and cumulative way (Singh and Dey, 2002).

Other explanations for financial assets' return persistence have to do with the importance of economic fundamentals (Nawrocki, 1995; Lo, 1991; Peters, 1989), and the use of

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[10] Hurst (1956) studied 76 natural phenomena. $\widehat{H}$ was significantly different from 0.5, and was close to 0.73 (σ = 0.092). Hurst found no evidence of significant autocorrelation in the first lags, which led him to reject short-term dependence as the source of this phenomenon; neither could he find a slow and gradual decay with increasing lags, which supported his rejection of long-term dependence in the traditional sense of Campbell *et al.* (1997).

[11] Mandelbrot and Wallis (1969a) were the first to recognize *R/S* as non-parametric, even in presence of extreme skewness or infinite variance. León and Vivas (2010), Martin *et al.* (2003), Willinger *et al.* (1999) and Peters (1996 and 1994) verified such statement.

privileged information (Menkens, 2007). Alternatively, some authors (Bouchaud *et al.*, 2008; Lillo and Farmer, 2004), based on the persistence of the number and volume of buying and selling orders in transactional systems, conclude that markets' liquidity make instantaneous trading impossible, leading to transactions' splitting, and decisions' clustering, resulting in market prices which don't fully reflect information immediately, but incrementally.

Concerning significance tests for $\widehat{H}$, two well-documented issues have to be taken into account (León and Vivas, 2010; Ellis, 2007; Couillard and Davison, 2005; Peters, 1994). First, there is a positive bias in the estimation of *H* resulting from finite time-series and minimum size of periods below approximately 1.000 observations. Second, $\widehat{H}$ for normal and non-normal distributed random variables distribute like a normal.

Regarding the first issue, the estimation bias resulting in the overestimation of $\widehat{H}$ can be conveniently assessed. Several assessment methods for estimating such bias have been documented, but this paper focuses on the single most well-known. First proposed by Anis and Lloyd (1976), subsequently revised by Peters (1994), and verified and applied by León and Vivas (2010), León and Reveiz (2011a), Leiton (2011), Ellis (2007) and Couillard and Davison (2005), the chosen method consists of a functional approximation for estimating the expected value of $(R/S)_n$ when the random variable is independent and of finite length. This method yields the expected Hurst exponent corresponding to an independent random variable, which will be noted as $\dot{H}$, and is based on the following calculation of the expected value of $(R/S)_n$:

$$E(R/S)_n = \frac{n-\frac{1}{2}}{n}\frac{1}{\sqrt{n\pi/2}}\sum_{i=1}^{n-1}\sqrt{\frac{n-i}{i}} \qquad \text{[F4]}$$

Any divergence of $\widehat{H}$ from $\dot{H}$ would signal the presence of long-term memory in time-series. However, as customary in statistical inference, it is critical to develop appropriate statistical tests to distinguish between significant and non-significant deviations from long-term independence null hypothesis.

The significance herein used test follows León and Vivas (2010) and León and Reveiz (2011a), and is similar to that proposed by Ellis (2007) and Couillard and Davison (2005). Because $\widehat{H}$'s distribution is established to be normal, even for random variables that are not, a conventional *t*-statistic test may be implemented. Let $\widehat{H}$ be the *R/S*'s estimated value of the Hurst exponent, $\hat{\mu}(\dot{H})$ and $\hat{\sigma}(\dot{H})$ the expected value and standard deviation of the expected Hurst exponent corresponding to an independent random variable ($\dot{H}$), the significance test would be as follows:[12]

[12] Let *N* be the length of time-series, due to $\widehat{H}$ distributing like a normal the ordinary choice for $\hat{\sigma}(\dot{H})$ is $\approx 1/N^{1/2}$ as in Peters (1994). According to Couillard and Davison (2005) this choice corresponds to infinite length time-series, and yields easy and frequent rejections of the independence null hypothesis. They propose $\approx 1/eN^{1/3}$, which is the authors choice.

$$t = \frac{\widehat{H} - \hat{\mu}(\dot{H})}{\hat{\sigma}(\dot{H})} \quad \text{[F5]}$$

As usual, if t is higher than ±1.96 it is possible to reject the null hypothesis of long-term independence with a 95% confidence level. The sign of t reveals the type of dependence: if it is positive (negative) there is evidence of persistence (antipersistence).

For convenience, given that $\dot{H}$ is the estimated Hurst exponent for random, independent and finite time-series of length N, the spread between $\dot{H}$ and 0.5 corresponds to the bias estimation resulting from using finite time-series and the choice of the size of periods *(n)*. Subtracting such spread from the Hurst exponent estimated using *R/S*, namely $\widehat{H}$, results in an adjusted estimated Hurst exponent, which will be noted as $\breve{H}$:

$$\breve{H} = \widehat{H} - \left(\dot{H} - 0.5\right) \quad \text{[F6]}$$

This adjusted estimated Hurst exponent ($\breve{H}$) is essential since it allows a practical and unbiased assessment of long-term dependence. All calculations implemented in the fourth section use this adjusted estimated Hurst exponent.[13]

3. Relaxing CAPM's long-term independence assumption

Like other Financial Theory milestones (e.g. Black & Scholes and Modern Portfolio Theory), CAPM relies on long-term independence of financial returns. Based on Greene and Fielitz (1979 and 1980), this section presents the CAPM's formulae resulting from acknowledging the existence of long-term dependence of financial returns, where the Hurst exponent (H) serves as the long-term dependence parameter.

a. Standard CAPM

Developed by Treynor, Sharpe, Lintner and Mossin in the 60's, CAPM has become the standard for stocks' valuation. Furthermore, from the estimated expected return, firm's capital cost can be calculated or portfolio performance can be assessed (Campbell *et al.*, 1997). In addition, CAPM provides a reference value to assess whether an action is undervalued or overvalued from its equilibrium value.

This model assumes that agents are risk averse and they maximize expected utility (in the von Neumann-Morgenstern sense) through investment decisions based on mean-variance criteria (Markowitz, 1959). Additionally, the model assumes market efficiency and perfect (no arbitrage opportunities), the absence of transaction costs and all investors sharing the

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[13] For this paper the adjusted estimated Hurst exponent is assumed to be stationary (i.e. returns follow an *unifractal* process). However, as pointed out by Di Mateo (2007), this exponent may vary though time, making the return process *multifractal*. The *unifractal* assumption is convenient since a reliable estimation of the Hurst exponent requires long time series, more than 1,000 daily observations according to authors' estimations and León and Vivas (2010).

same information. Deposit and loans rates are assumed to be equal, and it is assumed that money market resources are unlimited at the risk free rate. Also, it is assumed that there are homogeneous market expectations, which means that all agents have the same investment opportunities, and that they regard expected returns and variance in the same way.

Moreover, the model assumes that all investors have the same investment horizon, which relies on the existence of neutrality in the investment horizon. In the sense of Campbell *et al.* (1997), this means that it is necessary to assume that assets can be described as a random walk, in which returns are independent and identically distributed and jointly distributed as a multivariate normal.

As usual, let R_f be the risk free rate's return, R_m the market's return, R_j the risky asset's return, and $E(\cdot)$ the expectation operator, CAPM model yields that the equilibrium expected return of risky asset *j* is given by [F7].

$$E(R_j) = R_f + \beta_j[E(R_m) - R_f] \qquad \text{[F7]}$$

$$\beta_j = \frac{\sigma_{j,m}}{\sigma_m^2} \qquad \text{[F8]}$$

This relationship implies that equilibrium expected return of risky asset *(j)* is a linear and positive function of its risk premium, its covariance with the market ($\sigma_{j,m}$) and the market's variance (σ_m^2). In this sense, β_j is the portion of total risk that is priced, is the systematic[14] risk of risky asset *j*, also referred to as the market risk or non-diversifiable risk (Danthine and Donaldson, 2005), and is commonly known as the risky asset's beta. Note that when time horizon neutrality is assumed (i.e. under long-term independence of financial returns) choosing the holding period is irrelevant[15] and, therefore, the investment horizon is not specified in the model.

The unimportance of the holding period underneath the CAPM is based on the assumption of neutrality in the investment horizon, which also relies on the Brownian motion assumption. This implies that when returns are independent through time, then temporal consistency of volatility is valid, the square-root-of-time rule for scaling volatility can be used, and the investment horizon becomes irrelevant. However, in practice, financial time-series frequency and holding period is relevant due to their long-term dependence.

b. Adjusting CAPM for long-term dependence of financial returns

Under long-term dependence of financial returns some traditional methods reliant on Brownian motion or random walk theory become invalid. Due to findings supporting

[14] Please note that some authors refer to the non-diversifiable risk as systemic risk, instead of systematic risk.
[15] Notice that the classic CAPM formulation does not contains time sub-indexes.

dependence of returns, several measures of assets' risk and performance can vary through different time horizons. In particular, Greene and Fielitz (1979 and 1980) show how a generalized version of the well-known Brownian motion (i.e. fractional Brownian motion) may serve to adjust CAPM formulae in order to consider the presence of long-term dependence of financial returns.

According to Greene and Fielitz (1980), long-term dependent time series with Gaussian distributions may be modeled using the increments of a process called fractional Brownian motion. Fractional Brownian motion, denoted as $B_H(t)$, is a continuous random function which is a moving average of past increments of ordinary Brownian motion, traditionally denoted as $B(t)$, where the parameter H determines the direction and intensity of the dependence, as presented in the previous section.[16]

Let s be the size of the increments and let $\triangleq$ represent "identical in distribution", then fractional Brownian motion is a self-similar process that follows a s^H law for all t and s [F9], where the $H = 0.5$ case corresponds to customary Brownian motion[17], and to the celebrated square-root-of-time-rule.

$$B(t+s) - B(t) \triangleq s^H [B(t+1) - B(t)] \qquad \forall\, t, s \tag{F9}$$

Consequently, Greene and Fielitz (1979 and 1980) demonstrate that the variance and covariance of a fractional Brownian motion are a generalization of traditional Brownian motion. For two time series p and q, variances and covariance under a fractional Brownian motion are estimated as in [10 and 11] and [12], respectively; please note that when $H_p = H_q = 0.5$ the variance and covariance of a fractional Brownian motion correspond to those distinctive of Brownian motion.

$$Var\left[B_{H_p}(t+s) - B_{H_p}(t)\right] = s^{2H_p}\, Var[B_{H_p}(t+1) - B_{H_p}(t)] \tag{F10}$$

$$Var\left[B_{H_q}(t+s) - B_{H_q}(t)\right] = s^{2H_q}\, Var[B_{H_q}(t+1) - B_{H_q}(t)] \tag{F11}$$

$$\begin{aligned} &Cov\left\{\left[B_{H_p}(t+s) - B_{H_p}(t)\right], \left[B_{H_q}(t+s) - B_q(t)\right]\right\} \\ &= s^{H_p+H_q}\, Cov\left\{\left[B_{H_p}(t+1) - B_{H_p}(t)\right], \left[B_{H_q}(t+1) - B_{H_q}(t)\right]\right\} \end{aligned} \tag{F12}$$

Since the fractional Brownian motion estimation of variances and covariance differs from the standard Brownian motion, CAPM's measure of an asset's systematic risk (β_j) must be adjusted accordingly. Based on [F10, F11, F12], the required adjustment consists of using the Hurst exponents for the risky asset and the market (H_j and H_m) instead of relying on

[16] In this section we preserve Greene and Fielitz (1980) notation for the Hurst exponent (H), which is based on the traditional (non-adjusted) estimation of the Hurst exponent. Next section, which presents numerical results, uses the adjusted Hurst exponent ($\breve{H}$) for all calculations.

[17] In this sense, fractional Brownian motion is a generalization of geometric Brownian motion (León and Reveiz, 2011b).

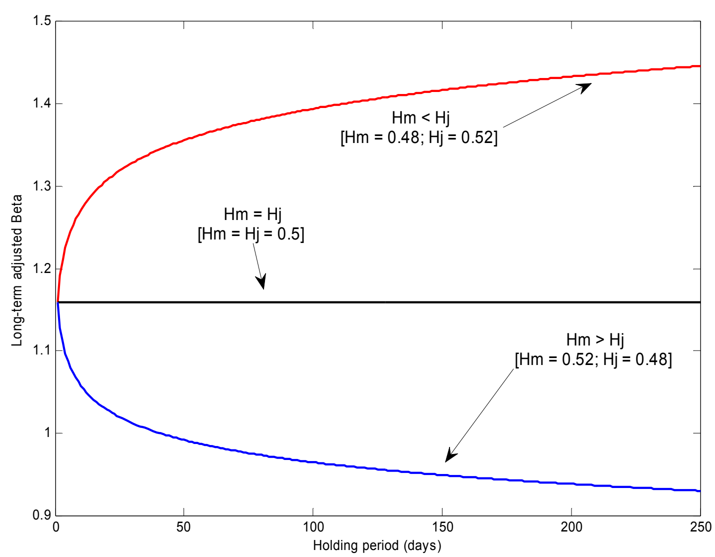
the customary ($H_j = H_m = 0.5$) assumption by Brownian motion. As exhibited in [13], this adjustment will result in an adjusted estimation of asset's systematic risk ($\check{\beta}_{j,s}$):

$$\check{\beta}_{j,s} = \frac{s^{(H_j+H_m)}\sigma_{j,m}}{s^{(2H_m)}\sigma_m^2} \quad \text{[F13]}$$

Unlike traditional CAPM's β_j, long-term dependence adjusted CAPM's $\check{\beta}_{j,s}$ varies with the size of the increment (s). This is, the systematic risk will be dependent on the investor's holding period, with two exceptions: (i) both processes are long-term independent ($H_m = H_j = 0.5$), or (ii) both processes share the same adjusted Hurst exponent ($H_m = H_j$).

As expected, in absence of the mentioned exceptions, the variation of $\check{\beta}_{j,s}$ throughout different holding periods will depend on the sign and magnitude of the inequality between H_m and H_j. From [F13] it is possible to infer that $\check{\beta}_{j,s}$ increases with *s* when $H_m < H_j$, and decreses when $H_m > H_j$. For three different scenarios ($H_m < H_j$; $H_m = H_j$; $H_m > H_j$), *ceteris paribus*[18], Figure 1 exhibits how $\check{\beta}_{j,s}$ deviates from β_j with increasing holding periods.

Figure 1
Impact of three scenarios of dependence on $\check{\beta}_{j,s}$
($H_m < H_j$; $H_m = H_j$; $H_m > H_j$)



Source: authors' calculations

According to Figure 1, assuming investment horizon neutrality ($H_m = H_j = 0.5$) when the market's and the risky asset's intensity of long-term dependence differs may underestimate or overestimate the investor's true exposure to systemic risk. Consequently,

[18] Figure 1 is constructed with $\sigma_{j,m} = 0.00021$ and $\sigma_m^2 = 0.00018$, corresponding to actual data for Apple Inc. and S&P100; data provided by Bloomberg.

standard CAPM's estimation of systemic risk may be misleading, where the severity of the problem increases with the holding period.[19]

Finally, based on the estimation of $\check{\beta}_{j,s}$, the estimation of expected return based on the long-term dependence adjusted CAPM is also a function of the size of the increment (s):

$$E\left(R_{j,s}\right) = R_f + \check{\beta}_{j,s}[E(R_m) - R_f] \qquad \text{[F14]}$$

Henceforth, as stressed by Greene and Fielitz (1980), in the presence of long-term dependence systematic risk estimates of the CAPM change with the differencing interval used to calculate returns. In this sense, as stressed by Holton (1992), absent a random walk risk becomes a two-dimensional quantity, with holding period almost as important a consideration as asset class. Consequently, standard performance measurement methods (i.e. Sharpe ratio, Treynor ratio, Jensen's alpha) are also subject to this differencing interval problem when long-term dependence is present. Moreover, as discussed in León and Reveiz (2011a) and León and Vela (2011), this problem also affects traditional asset allocation models based on mean-variance optimization, such as standard Modern Portfolio Theory or even the more elaborated Black-Litterman model.

Next section presents numerical results obtained when implementing the proposed adjustment to CAPM. Tests for the significance of the difference between standard and adjusted versions of CAPM are provided.

4. Numerical evidence of CAPM bias due to long-term dependence

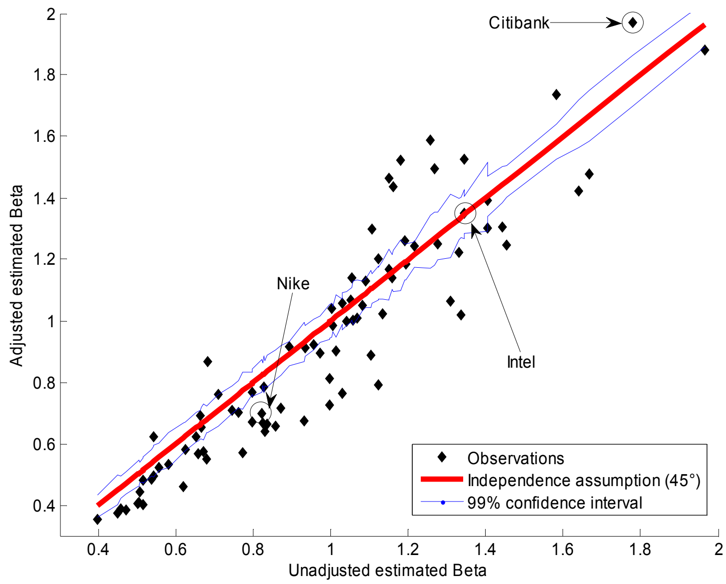
Based on the set of 85 stocks pertaining to the S&P100 with daily data from January 1st 2000 to June 21st 2012, this section presents numerical evidence of the bias resulting from overlooking long-term dependence of financial time series, along with the corresponding statistical significance. Figure 2 exhibits a scatter plot containing the observed relation between CAPM's customary estimation of β_j (x-axis) and the herein proposed long-term-dependency adjusted estimation of $\check{\beta}_{j,s}$ (y-axis), where the data set used has a daily frequency, the holding period (s) is assumed to be one year (i.e. 252 days), and the adjusted Hurst exponent ($\breve{H}$) is estimated as in [F6].

Because Figure 2 axis are symmetrical, if the long-term 1-year adjusted and unadjusted betas are equal (i.e. series are long-term independent or $\breve{H}_m = \breve{H}_j$) the observations would appear along the 45 degree thick line; conversely, observations diverging from the 45 degree solid line correspond to the case in which $\breve{H}_m \neq \breve{H}_j$. Based on the standard error of the estimation of each stock's unadjusted beta, thin lines above and below the 45 degree

[19] Based on the data used for Figure 1, the intensity of the error is about ±6.4% for a holding period equal to one week; ±12.0% for one month; ±22.3% for one year.

thick line represent the 99% confidence intervals; any divergence outside the region formed by the upper and lower intervals is statistically significant at 99%.

Figure 2
Comparing estimated β_j and $\breve{\beta}_{j,s}$
(85 stocks from S&P100, 252-day holding period, 99% confidence interval)



Source: authors' calculations.

According to authors' calculations, about 63.5% of the observations significantly diverge from the independence assumption: using a 99% confidence level, 45.9% of the observations are significantly below the independence assumption (i.e. unadjusted beta is overestimated), whilst 17.6% are significantly above the independence assumption (i.e. unadjusted beta is underestimated). Annex A displays the complete data set used to construct Figure 2.

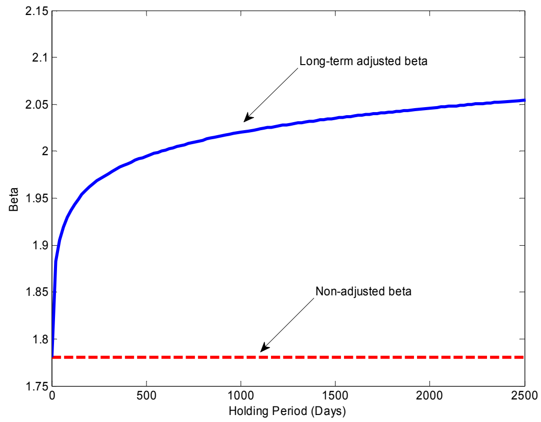
In order to examine the impact of these findings three stocks will be studied next. The three selected stocks are Citibank (C), Nike (NKE) and Intel (INTC). Since $\breve{H}_{S\&P100} = 0.4753$, $\breve{H}_{\mathrm{C}} = 0.4937$, $\breve{H}_{\mathrm{NKE}} = 0.4464$ and $\breve{H}_{\mathrm{INTC}} = 0.4760$, Citibank is representative of the case in which $\breve{H}_m < \breve{H}_j$, which results in an underestimated beta; Nike is representative of the case in which $\breve{H}_m > \breve{H}_j$, which results in an overestimated beta; Intel corresponds to the traditional CAPM's assumption where $\breve{H}_m \sim \breve{H}_j$ and where the long-term adjusted and unadjusted betas converge. Accordingly, Figure 3 confirms that CAPM's long-term independence assumption results in an increasing bias in the estimated parameter for Citibank and Nike, where the former's (latter's) consists of an increasing underestimation (overestimation) of the parameter; Intel's bias appears negligible.

Figure 3

$\breve{\beta}_{j,s}$ and β_j throughout holding horizons

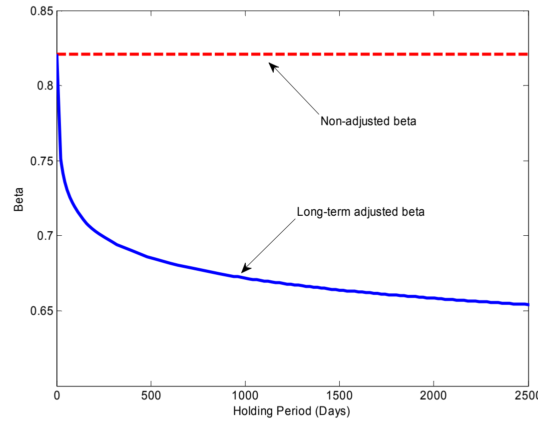
Citibank (C)

$\breve{H}_{\mathrm{C}} = 0.4937$



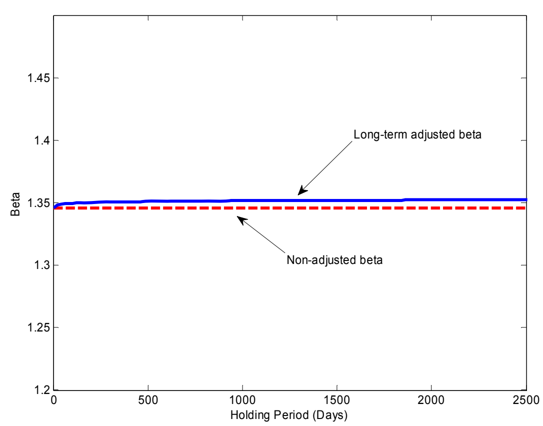
Nike (NKE)

$\breve{H}_{\mathrm{NKE}} = 0.4464$



Intel (INTC)

$\breve{H}_{\mathrm{INTC}} = 0.4760$



Source: authors' calculations.

Table 1 displays relevant information regarding the impact of long-term dependence on the three selected stocks for a one-year holding period. The second column ($\breve{H}$) corresponds to the adjusted Hurst exponent; third column corresponds to CAPM's customary estimation of β_j, with its standard error (σ_{β_j}) and R^2 in columns fourth and fifth. Sixth column presents the herein proposed [F13] long-term-dependency adjusted estimation of beta ($\breve{\beta}_{j,s}$). As expected, ignoring the long-term dependence results in CAPM underestimating the systemic risk of Citibank and overestimating Nike's, whereas the impact on Intel is null. A customary *t-statistic* significance test of the divergence between $\breve{\beta}_{j,s}$ and β_j is presented in the seventh column, with the null hypothesis ($\beta_j = \breve{\beta}_{j,s}$) being rejected for Citibank and Nike at a 99% confidence interval, whilst it is not rejected for Intel.

Table 1
Estimated 1-year holding period $\boldsymbol{\beta_j}$ and $\boldsymbol{\breve{\beta}_{j,s}}$
Citibank, Nike and Intel[a]

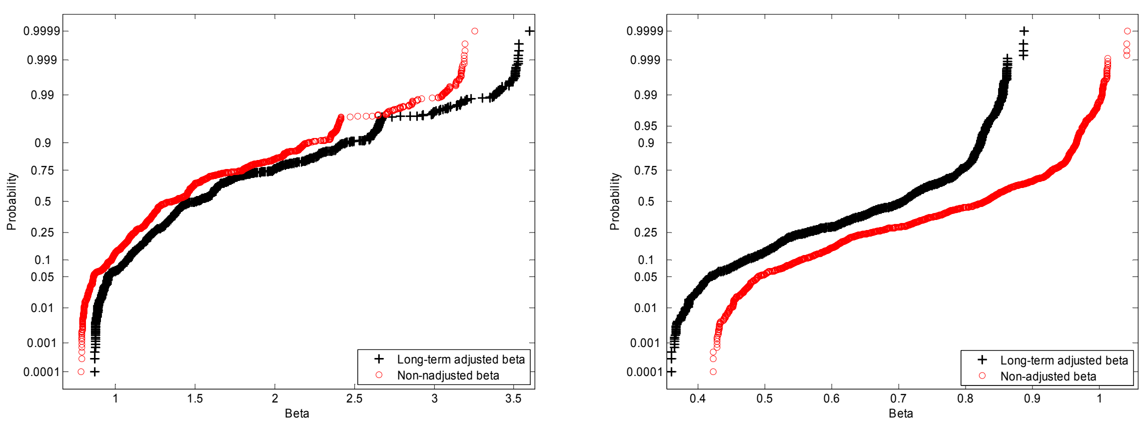
Stock	$\breve{H}$	β_j	σ_{β_j}	R^2	$\breve{\beta}_{j,s}$	$\dfrac{\beta_j - \breve{\beta}_{j,s}}{\sigma_{\beta_j}}$
C	0.49	1.78	0.04	0.45	1.97	-5.34 ***
INTC	0.48	1.35	0.03	0.46	1.35	-0.19
NKE	0.45	0.82	0.02	0.29	0.70	5.35 ***
S&P100	***0.48***	***1.00***	***N/A***	***N/A***	***1.00***	***N/A***

(***) significant at 99%; (**) significant at 95%; (*) significant at 90%.
[a] Calculations for the selected 85 stocks are included in Annex A.
Source: authors' calculations.

However, in order to avoid relying on –and inferring from- a single estimation of the $\breve{\beta}_{j,s}$ and β_j parameters, a bootstrap procedure (resampling with replacement) was implemented to approximate the empirical distribution of the parameters for the period under study.[20] In this sense, based on daily data for these three stocks, 5,000 estimations of 1-year holding period $\breve{\beta}_{j,s}$ and β_j were made from an equal number of resampled 252-days[21] windows. Such procedure results in the simulated density function of both parameters for each stock, for a 1-year holding period. Figure 4 displays the probability plot corresponding to each simulated density function.

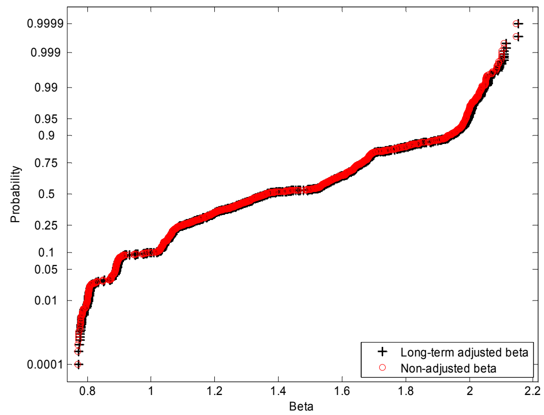
Figure 4
Probability plots of simulated density functions of $\breve{\beta}_{j,s}$ and β_j

Citibank (C) $\breve{H}_{\mathrm{C}} = 0.4937$ — Nike (NKE) $\breve{H}_{\mathrm{NKE}} = 0.4464$



[20] As acknowledged by Dowd (2005), the main purpose of bootstrap is to assess the accuracy of parameter estimates, where a new –larger- sample is built from the original data set, without depending on potentially unreliable assumptions such as normality or the existence of large samples.
[21] Customary usage of CAPM usually consists of using last year of daily observations (i.e. 252 days approximately) in order to make estimations for 1-year holding horizon. Using other lengths (e.g. 100 and 500 days) did not affect the analysis.

Intel (INTC)
$\breve{H}_{\text{INTC}} = 0.4760$



Source: authors' calculations.

The density function confirms that ignoring long-term dependence of time series results in CAPM's overestimation of the parameter for Citibank, with the long-term adjusted beta dominating the non-adjusted beta in the probability plot; this is, $\breve{\beta}_{j,s} \geq \beta_j$ for any probability level. Likewise, the density function confirms that CAPM overestimates Nike's beta, where $\breve{\beta}_{j,s} \leq \beta_j$ for any probability level. As expected, unlike the case of Citibank and Nike, both Intel's density functions, long-term adjusted and non-adjusted, appear to be identical.

Despite the mere inspection of the probability plots for these three stocks seems conclusive, Table 2 provides two non-parametric tests of distributional equality.[22] The first test, two-sample Kolmogorov-Smirnov, tests the null hypothesis that both samples' are from the same continuous distribution; the second, the Mann-Whitney U, tests the null hypothesis that both samples are independent samples from identical continuous distributions with equal medians.[23]

[22] Non-parametric tests are required since standard tests (i.e. Jarque-Bera, Lilliefors and Kolmogorov-Smirnov) rejected the null hypothesis of normality of the observations pertaining to the simulated density functions.
[23] The Kolmogorov-Smirnov test compares the empirical distribution of two samples, where the statistic is calculated as the maximum absolute difference between the two distributions. The U test of Mann-Whitney is defined as the number of times that Y preceeds X, where X and Y are sorted in an ascended way. Both tests are described in Gibbons and Chakraborti (2003).

Table 2
Estimated 1-year holding period $\boldsymbol{\beta_j}$ and $\boldsymbol{\breve{\beta}_{j,s}}$
Citibank, Nike and Intel[a]

Stock	$\breve{H}$	β_j	σ_{β_j}	R^2	$\breve{\beta}_{j,s}$	$\frac{\beta_j - \breve{\beta}_{j,s}}{\sigma_{\beta_j}}$	*K-S Test*[b]	*M-W Test*[c]
C	0.49	1.78	0.04	0.45	1.97	-5.34 ***	0.18 ***	15,7 ***
INTC	0.48	1.35	0.03	0.46	1.35	-0.19	0.02	1,1
NKE	0.45	0.82	0.02	0.29	0.70	5.35 ***	0.45 ***	-38,5 ***
S&P100	***0.48***	***1.00***	***N/A***	***N/A***	***1.00***	***N/A***	***N/A***	***N/A***

(***) significant at 99%; (**) significant at 95%; (*) significant at 90%.
[a] Calculations for the selected 85 stocks is included in Annex A.
Source: authors' calculations. Source: authors' calculations. Data from Bloomberg.

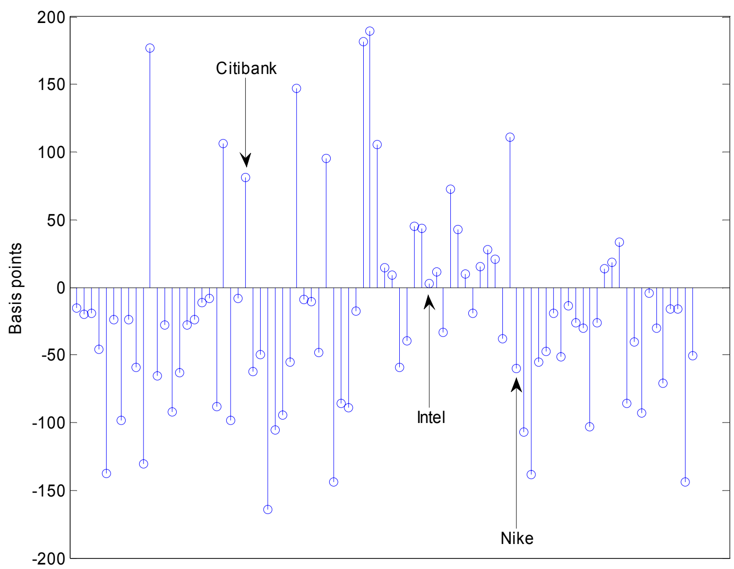
Table 2 shows that both tests reject the null hypothesis corresponding to the simulated distributions of $\breve{\beta}_{j,s}$ and β_j belonging to identical continuous distributions for Citibank and Nike; as before, Intel's null hypothesis can't be rejected. This confirms visual inspection of the simulated density function, along with standard *t-statistic* of the difference between estimated $\breve{\beta}_{j,s}$ and β_j.

As presented in Annex A, the three tests herein implemented confirm that most of the selected 85 stocks exhibit significant bias in the estimation of their beta parameter under the assumption of long-term independence. At a 99% confidence level, the conventional *t-statistic*, the Kolmogorov-Smirnov and the Mann-Whitney tests, they reject the null hypothesis of identical betas after adjusting for long-term dependence for 63.5%, 98.82% and 97.65% of the stocks, respectively.

Consequently, as a result of the aforementioned estimation bias, it is fair to anticipate that expected returns should also be biased.[24] Based on the long-term adjusted and unadjusted standard equation of the CAPM [F14 and F7, respectively], let the risk free rate equal the 1-year rate extracted from the U.S. Treasury zero coupon curve ($R_f = 0.23\%$); the expected return of the market equal the annualized mean return of the last year of data of the S&P100 ($E(R_m) = 5.37\%$); the $\breve{\beta}_{j,s}$ and β_j equal those reported in Annex 2, Figure 5 exhibits the difference between the long-term adjusted ($E(R_{j,s})$) and unadjusted ($E(R_j)$) expected returns for each of the 85 stocks selected.[25]

[24] Please note that significance tests were conducted based on the divergence between $\breve{\beta}_{j,s}$ and β_j since these parameters require less user-defined inputs than the resulting expected returns.
[25] The risk free rate and the expected market return are calculated as of 21st June 2012.

Figure 5
Comparing estimated expected returns ($E(R_{j,s}) - E(R_j)$)
(85 stocks from S&P100, 252-day holding period, in basis points)



Positive results correspond to the $E(R_{j,s}) > E(R_j)$ case (i.e. traditional CAPM underestimates expected return)
Source: authors' calculations.

Since the difference is calculated as ($E(R_{j,s}) - E(R_j)$), positive (negative) results correspond to the underestimation (overestimation) of the expected return under the long-term independence assumption. In this sense, based on the herein proposed investment horizon dependent CAPM, the 1-year equilibrium expected return for Citibank is 81.84bps higher than traditional CAPM estimates; for Nike it is 60.30bps lower, and for Intel 2.63bps higher. For the whole sample (85 stocks) the mean absolute bias is about 59.07bps.

5. Final remarks

It is obvious that an investor will always regard his investment horizon as a relevant input for designing his portfolios; likewise, it is also evident that a financial officer will consider an assets' holding period for making investment decisions. However, the CAPM, which is considered the most successful and enduring model for asset pricing and returns forecasting, disregards time (i.e. the investment horizon) as an important input; this results in market risk –the beta- being constant over any holding period.

As forewarned by Holton (1992), after relaxing the long-term independence assumption risk becomes a two-dimensional quantity, with holding period almost as important a consideration as asset class. For the CAPM, Greene and Fielitz (1980) alert that in the presence of long-term dependence systematic risk estimates of the CAPM –the betas- change with the differencing interval used to calculate returns. Based on numerical evidence provided in this paper, worries by Holton and Green and Fielitz are genuine and statistically significant: according to three tests herein implemented with a 99% confidence level, more than 60% of the stocks under analysis exhibit significantly different beta parameters, and the resulting equilibrium expected return exhibits a $\pm$60bps bias for a contemporary one-year investment horizon.

Because of the significance of the results and the importance of the CAPM for theoretical and practical purposes, the herein proposed procedure to adjust betas for long-term dependence may provide investors, asset managers and financial officers a parsimonious method to relax the long-term independence assumption and attain a convenient investment horizon dependent CAPM. As quantitatively demonstrated, the relevance of the adjustment increases with the investment horizon; therefore, long-term investors or asset managers (e.g. pension funds, sovereign wealth managers, and foreign reserves managers) will profit from employing a CAPM model where the holding period is significant.

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7. Annex A

Estimated 1-year holding period β_j and $\breve{\beta}_{j,s}$ for selected S&P100 stocks[a]

Stock	$\breve{H}$	β_j	σ_{β_j}	R^2	$\breve{\beta}_{j,s}$	$\frac{\beta_j - \breve{\beta}_{j,s}}{\sigma_{\beta_j}}$	*K-S Test*[b]	*M-W Test*[c]
DELL	0.51	1.16	0.03	0.35	1.44	-9.65 ***	0.54 ***	49,5 ***
GD	0.52	0.68	0.02	0.28	0.87	-9.59 ***	0.49 ***	39,4 ***
F	0.52	1.18	0.04	0.26	1.52	-9.46 ***	0.53 ***	53,3 ***
FCX	0.52	1.26	0.04	0.27	1.59	-9.01 ***	0.41 ***	29,0 ***
CAT	0.50	1.11	0.02	0.46	1.30	-8.94 ***	0.30 ***	32,3 ***
AAPL	0.52	1.15	0.04	0.25	1.46	-8.88 ***	0.38 ***	43,2 ***
NOV	0.51	1.27	0.04	0.28	1.50	-6.25 ***	0.29 ***	22,9 ***
JPM	0.49	1.58	0.02	0.56	1.74	-6.18 ***	0.28 ***	25,1 ***
C	0.49	1.78	0.04	0.45	1.97	-5.34 ***	0.18 ***	15,7 ***
EMC	0.50	1.35	0.03	0.33	1.52	-5.21 ***	0.39 ***	26,8 ***
LMT	0.50	0.54	0.02	0.17	0.62	-3.77 ***	0.42 ***	32,5 ***
HD	0.49	1.05	0.02	0.41	1.14	-3.69 ***	0.30 ***	29,5 ***
HON	0.49	1.12	0.02	0.45	1.20	-3.63 ***	0.29 ***	27,0 ***
TWX	0.49	1.19	0.03	0.41	1.26	-2.68 ***	0.24 ***	15,5 ***
MRK	0.49	0.71	0.02	0.25	0.76	-2.39 ***	0.24 ***	25,2 ***
MSFT	0.48	1.09	0.02	0.48	1.13	-2.01 **	0.17 ***	16,0 ***
GE	0.48	1.22	0.02	0.59	1.24	-1.64 **	0.10 ***	8,2 ***
TGT	0.48	1.00	0.02	0.36	1.04	-1.52 *	0.14 ***	11,3 ***
MDT	0.48	0.66	0.02	0.26	0.69	-1.48 *	0.16 ***	11,5 ***
IBM	0.48	0.89	0.02	0.45	0.91	-1.23	0.13 ***	8,4 ***
SBUX	0.48	1.03	0.03	0.33	1.06	-0.99	0.10 ***	8,4 ***
LOW	0.48	1.05	0.02	0.39	1.07	-0.76	0.09 ***	7,6 ***
HAL	0.48	1.15	0.04	0.25	1.17	-0.52	0.04 ***	2,9 ***
INTC	0.48	1.35	0.03	0.46	1.35	-0.19	0.02	1,1
USB	0.47	1.19	0.03	0.41	1.18	0.38	0.04 ***	-2,4 **
CSCO	0.47	1.41	0.03	0.45	1.39	0.56	0.09 ***	-4,6 ***
BMY	0.47	0.67	0.02	0.22	0.65	0.65	0.06 ***	-4,4 ***
DOW	0.47	1.16	0.03	0.40	1.14	0.69	0.08 ***	-5,5 ***
QCOM	0.47	1.28	0.03	0.32	1.25	0.75	0.06 ***	-4,7 ***
BA	0.47	0.93	0.02	0.37	0.91	0.94	0.07 ***	-6,1 ***
DD	0.47	1.01	0.02	0.49	0.99	1.11	0.15 ***	-8,4 ***
WAG	0.47	0.65	0.02	0.24	0.62	1.35 *	0.16 ***	-9,7 ***
DIS	0.47	1.08	0.02	0.47	1.05	1.48 *	0.12 ***	-10,3 ***
FDX	0.47	0.96	0.02	0.39	0.92	1.50 *	0.11 ***	-8,6 ***
MCD	0.46	0.56	0.02	0.21	0.52	1.68 **	0.12 ***	-9,5 ***
AMGN	0.47	0.83	0.03	0.25	0.78	1.72 **	0.13 ***	-11,9 ***
ABT	0.46	0.51	0.02	0.19	0.48	1.79 **	0.15 ***	-13,6 ***
BHI	0.47	1.06	0.03	0.27	1.00	1.79 **	0.12 ***	-8,2 ***
ALL	0.47	1.04	0.02	0.38	1.00	1.81 **	0.09 ***	-6,7 ***
PFE	0.47	0.74	0.02	0.33	0.71	1.81 **	0.19 ***	-16,0 ***
MMM	0.47	0.80	0.02	0.45	0.77	1.82 **	0.12 ***	-10,8 ***
RTN	0.46	0.58	0.02	0.16	0.53	1.90 **	0.23 ***	-20,8 ***
WMB	0.46	1.40	0.05	0.21	1.30	2.13 **	0.23 ***	-13,1 ***

AEP	0.46	0.62	0.02	0.23	0.58	2.14 **	0.18 ***	-16,7 ***
SLB	0.47	1.07	0.03	0.33	1.01	2.18 **	0.22 ***	-11,1 ***
BAX	0.46	0.54	0.02	0.16	0.49	2.18 **	0.16 ***	-15,8 ***
MS	0.47	1.96	0.03	0.55	1.88	2.62 ***	0.18 ***	-12,3 ***
SO	0.45	0.40	0.02	0.17	0.35	2.75 ***	0.24 ***	-16,0 ***
BRK/B	0.46	0.54	0.02	0.22	0.48	2.93 ***	0.17 ***	-12,7 ***
VZ	0.46	0.76	0.02	0.34	0.70	3.07 ***	0.18 ***	-17,2 ***
MO	0.44	0.45	0.02	0.13	0.38	3.55 ***	0.33 ***	-24,0 ***
ORCL	0.46	1.33	0.03	0.39	1.22	3.61 ***	0.18 ***	-17,7 ***
APC	0.45	1.01	0.03	0.27	0.90	3.82 ***	0.18 ***	-13,2 ***
CVS	0.45	0.67	0.02	0.19	0.57	3.88 ***	0.35 ***	-29,7 ***
HPQ	0.46	1.13	0.03	0.36	1.02	4.09 ***	0.22 ***	-21,8 ***
UTX	0.46	0.97	0.02	0.47	0.90	4.19 ***	0.35 ***	-34,3 ***
LLY	0.45	0.66	0.02	0.24	0.57	4.30 ***	0.32 ***	-25,8 ***
JNJ	0.45	0.51	0.01	0.27	0.44	4.33 ***	0.30 ***	-26,9 ***
HNZ	0.45	0.46	0.02	0.22	0.39	4.49 ***	0.37 ***	-30,4 ***
PG	0.44	0.50	0.02	0.19	0.41	5.13 ***	0.31 ***	-30,6 ***
COF	0.45	1.67	0.04	0.41	1.48	5.31 ***	0.27 ***	-23,0 ***
NKE	0.45	0.82	0.02	0.29	0.70	5.35 ***	0.45 ***	-38,5 ***
AMZN	0.44	1.31	0.04	0.21	1.06	5.43 ***	0.29 ***	-32,0 ***
PEP	0.44	0.47	0.02	0.22	0.39	5.45 ***	0.46 ***	-42,9 ***
CL	0.44	0.50	0.02	0.20	0.40	5.47 ***	0.45 ***	-44,0 ***
BK	0.46	1.44	0.03	0.51	1.31	5.47 ***	0.37 ***	-27,4 ***
T	0.44	0.80	0.02	0.34	0.67	6.45 ***	0.29 ***	-30,9 ***
BAC	0.45	1.64	0.03	0.43	1.42	6.51 ***	0.34 ***	-26,7 ***
KO	0.43	0.51	0.02	0.23	0.40	6.66 ***	0.54 ***	-48,0 ***
WMT	0.44	0.68	0.02	0.31	0.55	7.10 ***	0.43 ***	-39,4 ***
UNP	0.44	0.87	0.02	0.37	0.71	7.65 ***	0.45 ***	-35,5 ***
COST	0.43	0.84	0.02	0.30	0.66	7.66 ***	0.47 ***	-48,9 ***
NSC	0.44	1.00	0.02	0.35	0.81	7.72 ***	0.49 ***	-47,8 ***
UNH	0.42	0.77	0.03	0.22	0.57	7.73 ***	0.31 ***	-33,9 ***
EXC	0.42	0.62	0.02	0.23	0.46	7.82 ***	0.34 ***	-32,5 ***
SPG	0.44	1.10	0.03	0.37	0.89	8.37 ***	0.23 ***	-17,0 ***
APA	0.42	0.93	0.03	0.26	0.67	9.28 ***	0.44 ***	-30,8 ***
AXP	0.45	1.45	0.02	0.58	1.25	9.36 ***	0.36 ***	-32,6 ***
XOM	0.44	0.83	0.02	0.44	0.67	9.42 ***	0.44 ***	-39,7 ***
COP	0.43	0.86	0.02	0.35	0.66	9.60 ***	0.50 ***	-35,0 ***
CVX	0.43	0.83	0.02	0.42	0.64	10.75 ***	0.55 ***	-38,8 ***
OXY	0.42	1.00	0.02	0.35	0.72	11.24 ***	0.53 ***	-34,1 ***
WFC	0.43	1.34	0.03	0.42	1.02	11.37 ***	0.35 ***	-33,2 ***
CMCSA	0.41	1.12	0.02	0.40	0.79	13.60 ***	0.68 ***	-65,6 ***
EMR	0.42	1.03	0.02	0.50	0.76	14.62 ***	0.78 ***	-71,9 ***
S&P100	***0.48***	***1.00***	***N/A***	***N/A***	***1.00***	***N/A***	***N/A***	***N/A***

[a] S&P100's stocks with missing data for the analyzed period (January 1st 2000 – June 21st 2012) were discarded. Remaining stocks (85) are ordered according to the significant test on seventh column. The 1-year treasury zero coupon rate was used as the risk free rate. The expected return of the market corresponds to the annualized average daily log-return of the S&P100.

(***) significant at 99%; (**) significant at 95%; (*) significant at 90%.

Source: authors' calculations.