Identifiability of a coincident index model for the Colombian economy

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Abstract

In this theoretical report, the identifiability property of a coincident index model is studied. As a result, characterization of the identifiability conditions solves a model specification problem, which was detected in the design of an earlier index for the Colombian economy.

Key words and phrases. Coincident economic index, model identifiability, state space model.

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1 Introduction

Nieto and Melo (2001) developed a methodology for computing a coincident index in levels for the economic activity, which is based on previous work of Stock and Watson (1989, 1991, 1992). Their results were then applied by Melo *et al.* (2001) in the design of a coincident index for the Colombian economy and some of the index properties were analyzed.

Although the Colombian coincident index obtained by Melo *et al.* (2001) was adequate for tracking the state of the economy and some properties of the state space model considered for designing the coincident index were studied by Nieto and Melo (2001), a model specification problem was detected by those authors. It consisted in the empirical persistence of some assumed stationary processes and the strong difficulty of the optimization routines for finding the likelihood function maximum. I feel these problems have to deal with the model *identifiability*, an issue that has not been addressed yet, on the knowledge of the present author. This is a key characteristic that must be taken into account in model fitting, in order to get a precise idea about the behavior of the likelihood function and, in turn, about the precision and accuracy of the estimated parameters and about the model specification.

This report is a summary of the main results found by Nieto (2002) about this identifiability topic. In Section 2 I present the main equations that define the coincident-index model and in Section 3, I include some of the Nieto's (2002) basic results. The last section concludes.

2 The statistical model

The basic hypothesis for the construction of a coincident index for the so-called state of the economy process, denoted by $\{C_t\}$, is the following: there are observable processes $\{X_{1t}\}$,...,$\{X_{nt}\}$, each one integrated of order one and called coincident processes, that have a contemporaneous relationship with $\{C_t\}$ given by the equation

$$X_{it} = \beta_{it} + \gamma_i C_t + u_{it} \ ,$$

for all $t = 1, ..., N$, N the length of the sample period, and for all $i = 1, ..., n$, where β_{it} is a deterministic component that can include seasonal components, γ_i is a constant that represents the weight of C_t in X_{it} and u_{it} is a stochastic component inherent to X_{it} and independent of C_t, which follows the stationary autoregressive process

$$D_i(B) u_{it} = \epsilon_{it} \ ,$$

where $D_i(B) = 1 - d_{i1}B - ... - d_{ik}B^k$, with B as the lag operator and $\{\epsilon_{it}\}$ a Gaussian white noise process with variance σ_i^2. We also assume that the stochastic processes $\{\epsilon_{it}\}$ are mutually independent among them, which implies the mutual independence of the $\{u_{it}\}$ processes. Essentially, the previous equations express that a deseasonalized coincident variable is a linear transformation of the state of the economy (the latent variable) plus an intrinsic random noise. A difference with previous methodologies is that the eventual seasonal component in the observable variables is included directly into the relation between X_{it} and C_t, which has some advantages as quoted by Nieto and Melo (2001).

The stochastic dynamic of $\{C_t\}$ is described by the model

$$\phi(B)\Delta C_t = \delta + \eta_t \ ,$$

where $\phi(B)$ is an autoregressive stationary operator of order p, δ is a constant, and $\{\eta_t\}$ is a Gaussian white noise process with variance σ_η^2. This equation shows another essential assumption of the methodology: $\{C_t\}$ is an integrated process of order 1 [I(1)]. Let $\mathbf{X}_t = (X_{1t}, ..., X_{nt})'$, $\beta_t = (\beta_{1t}, ..., \beta_{nt})'$, $\gamma = (\gamma_1, ..., \gamma_n)'$, $\mathbf{u}_t = (u_{1t}, ..., u_{nt})'$ and $\epsilon_t = (\epsilon_{1t}, ..., \epsilon_{nt})'$, then the previous equations can be rewritten in the following vectorial form:

$$\mathbf{X}_t = \beta_t + \gamma C_t + \mathbf{u}_t \tag{1}$$

$$\phi(B)\Delta C_t = \delta + \eta_t \tag{2}$$

$$D(B)\mathbf{u}_t = \epsilon_t \tag{3}$$

where $D(B) = I - D_1 B - ... - D_k B^k$, with I the identity matrix of order n, and $D_i = \text{diag}\{d_{1i}, ..., d_{ni}\}$. As noted by Nieto and Melo (2001), this specification of the coincident model implies that the process $\{\mathbf{X}_t\}$ is cointegrated.

The statistical problem consists in estimating C_t using the observed information up to time t; $t = 1, ..., N$. The estimated process, $\{C_{t|t} : t = 1, ..., N\}$ say, is considered as the *coincident index*. Technically, it means to compute $C_{t|t} = E(C_t | \mathbf{X}_1, ..., \mathbf{X}_t)$, $t = 1, ..., N$. We can use the Kalman filter to obtain these conditional expected values, and the corresponding details were developed by Nieto and Melo (2001) and applied by Melo *et al.* (2001). Computation of $C_{t|t}$ is based on the so-called model hyperparameters, which must be estimated; hence, in practice one gets an estimate $\{\hat{C}_{t|t}\}$ of $\{C_{t|t}\}$. Thus, before addressing the parameter estimation part of the problem, one needs to know about the model identifiability.

3 Model identifiability

Nieto (2002) has obtained some results about this topic on the basis of Bickel and Docksum's (1977) identifiability definition. This means to consider the joint probability density of the process $\{\mathbf{X}_t : t = 1, ..., N\}$, which is assumed to be parameterized by a vector ψ in some Euclidian parameter space. Let $p(\mathbf{x}, \psi)$ be this parameterized density where $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_N)$. There is not identifiability of the process distribution when ψ_1 is not equal to ψ_2 and $p(\mathbf{x}, \psi_1) = p(\mathbf{x}, \psi_2)$ for all $\mathbf{x}$ in the range of $\mathbf{X} = (X_1, ..., X_N)$. Equivalently, the joint distribution is identifiable if and only if each marginal distribution is identifiable.

With that concept in mind, I begin exploring this property for Nieto and Melo's (2001) state space model by means of some theoretical simple cases. In the first one, I consider a single coincident variable X and set $\beta_t = 0$, $p = 0$, $\delta = 0$, and $k = 1$. Then, the basic equations are given by

$$\begin{aligned} X_t &= \gamma C_t + u_t \\ \Delta C_t &= \eta_t \\ (1 - dB)u_t &= \epsilon_t \ , \end{aligned}$$

where γ is a real number and $|d| < 1$. As usual, I set $\sigma_\eta^2 = 1$, which gives a model reparameterization in order to get C_t dimensionless. I shall examine the identifiability character of the X_t's marginal density, for each $t = 1, 2, ..., N$.

The equivalent state space model for the previous three equations is given by

$$\boldsymbol{\alpha}_t = T\boldsymbol{\alpha}_{t-1} + R\boldsymbol{\zeta}_t \ ,$$

as the system equation, and

$$X_t = Z\boldsymbol{\alpha_t}$$

as the observation equation, where $\boldsymbol{\alpha_t} = (C_t, u_t)'$, $\boldsymbol{\zeta_t} = (\eta_t, \epsilon_t)'$, $Z = [\gamma, 1]$, $T = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Additionally, I note that $\mathrm{E}(\boldsymbol{\zeta_t}\boldsymbol{\zeta_t}') = Q = \begin{bmatrix} 1 & 0 \\ 0 & \sigma_1^2 \end{bmatrix}$.

Here, the vector of hyperparameters is given by $\psi = (\gamma, d, \sigma_1^2)$.

It is easily seen that

$$X_t = (ZT^t)\boldsymbol{\alpha_0} + \sum_{j=0}^{t-1} ZT^j R\boldsymbol{\zeta_{t-j}} \ ,$$

and because of the form of the matrices Z, T, R, and Q, I obtain that

$$X_t = \gamma\alpha_{01} + d^t\alpha_{02} + \sum_{j=0}^{t-1}(\gamma\eta_{t-j} + d^j\epsilon_{t-j})$$

with $\boldsymbol{\alpha_0} = (\alpha_{01}, \alpha_{02})'$. Hence, $X_t \sim N(\mu_t, \sigma_t^2)$ where $\mu_t = \gamma\alpha_{01} + d^t\alpha_{02}$ and $\sigma_t^2 = t\gamma^2 + \sigma_1^2 \sum_{j=0}^{t-1}(d^2)^j$. I observe immediately that for $\pm d$ and $t = 2m$, for some positive integer m, the corresponding densities are the same and $\psi_1 = (\gamma, d)$ is different of $\psi_2 = (\gamma, -d)$. That is to say, there is not identifiability of the X_t's distribution for $t = 2m$; therefore, the joint distribution of $\{X_t\}$, or equivalently the statistical model (4)-(5), is not identifiable.

The above finding suggests reparameterizing the model or to fix some parameters, in order to get identifiability. If I use the second approach, a possible parameter to be restricted is d; however, even if d is fixed but I put $\alpha_{01} = 0$, one finds that there is also no identifiability with respect to γ

because for $\pm\gamma$, I can obtain the same density. Thinking in real applications to the economy, I recommend fixing d and to restrict γ to be nonnegative. It is interesting to note that a negative value of γ provides the same information that its additive inverse, about the coincident relationship between X_t and C_t.

To understand both the analytical and geometrical behavior of the unrestricted likelihood function for this model, I simulated it with the following parameter values: $\gamma = 1$, $d = 0.7$, and $\sigma_1^2 = 1$. As is well known, identifiability problems can induce ill-behavior in the likelihood function, which is reflected in the appearance of several local maxima or flat parts in its geometrical representation, which is a hypersurface. The log-Gaussian likelihood function for the model is given by

$$l(\psi) = -0.5\log 2\pi - 0.5\sum_{t=1}^{N}\log f_t - 0.5\sum_{t=1}^{N}\nu_t^2/f_t \ ,$$

with ν_t the one-step-ahead prediction error at t, and f_t its mean squared error. Its computation is carried out using the Kalman filter. Then, for different fixed values of d in the interval $(-1,1)$ and fixing $\sigma_1^2 = 1$, which is not a loss in generality, I obtained the corresponding likelihood functions for discrete values of γ in the interval $(-10,10)$, which are chosen with a step of 0.1. In Figure 1, I present the graphical results where the point 101 in the horizontal axis corresponds to $\gamma = 0$, 102 to $\gamma = 0.1$, 100 to $\gamma = -0.1$, and so on. The curves presented there correspond to the following values of d: -0.7, -0.3, 0.1, 0.6, and 0.8. As one can see, all the functions attained their two local maximum values at the same values for γ, which are approximately 0.4 (or 105) and -0.4 (or 97). The same observation holds for another values of d. Overall, the likelihood functions have two local maxima that are attained at the same values of γ irrespective of d. Obviously, one can invert the situation and fixing γ, let d vary in the interval $(-1, 1)$ for finding the likelihood function profiles; however, the results are similar. In Figure 2, I present the 2-dimensional likelihood-function surface, keeping σ_1^2 constant, with some contours (level curves), where one can see that there is an enlarged hill in the d direction and above the line $\gamma = 1$, approximately. But most important, the contours indicate that there is not absolute maximum on the parameter space. All of these geometrical observations are in accordance with the theoretical findings above.

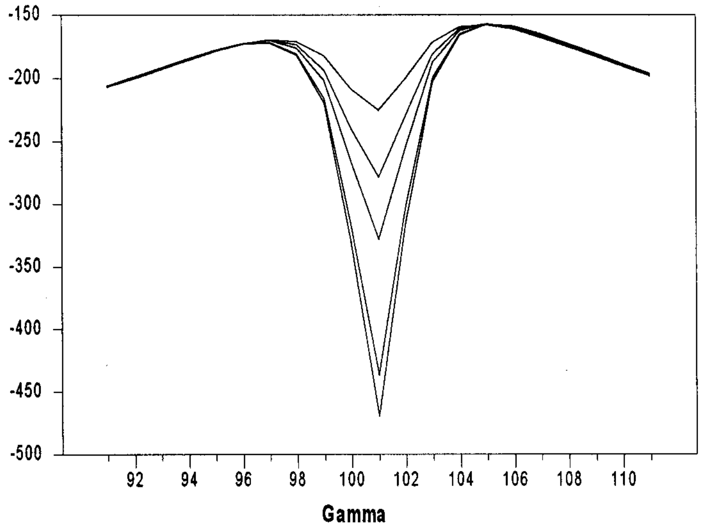

Figure 1: Likelihood functions for γ that correspond to different fixed values of d

Simulations for this model with $k \geq 2$ were also conducted and the identifiability results were the same, i.e. the model is not identifiable. In order to

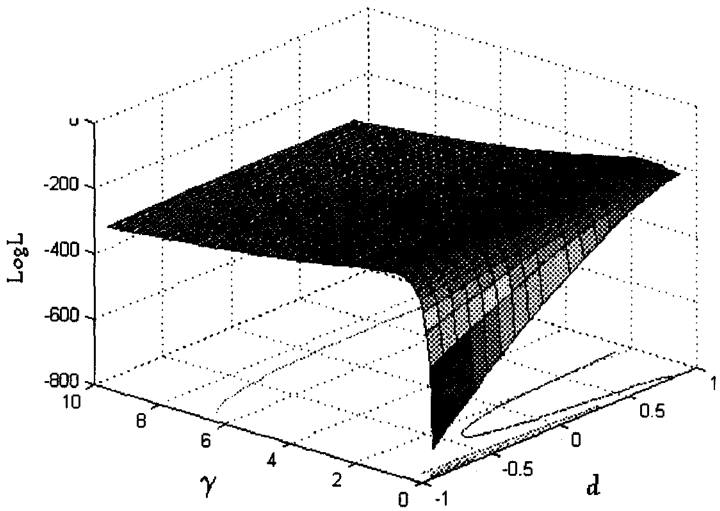

Figure 2: Likelihood surface for the first simple model when γ is restricted to be nonnegative

take into account the autoregressive parameters of the latent process $\{C_t\}$, I simulated again a univariate model with $\beta_t = 0$, $\gamma = 1$, $k = 1$, $d = 0.7$, $p = 1$, $\phi_1 = 0.5$, and $\sigma_1^2 = 1$. Fixing σ_1^2, the main parameter vector is (γ, d, ϕ_1), which I concentrate the analysis on. I plotted the three possible likelihood surfaces, each one corresponding to a pair of parameters, with their respective contours in Figures 3-5 and, as one can see there, there is indication of no identifiability.

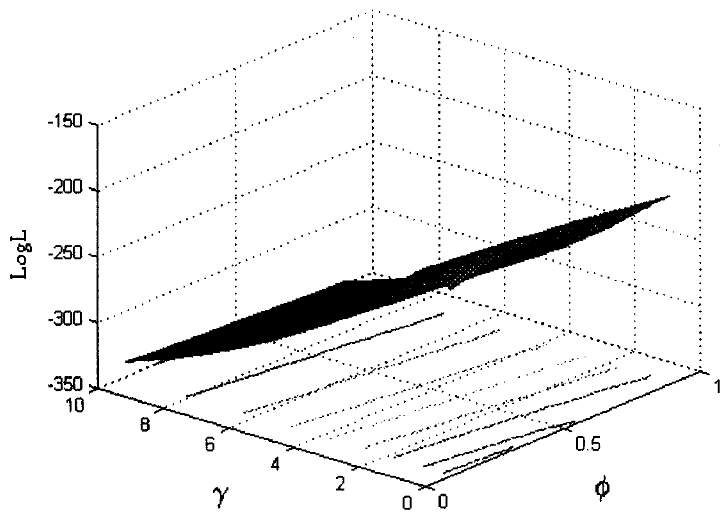

Figure 3: Likelihood surface for the second model with d fixed.

The graphs signal the following important facts: (i) Fixing either d or ϕ_1, the likelihood surface tends to have large flat parts suggesting that the likelihood function either does not have absolute maximum or *does* have but it is very difficult to detect via a numerical optimization procedure. (ii) Fixing γ, the surface for d and ϕ_1 is well-behaved in the sense the likelihood function has only one maximum on the parameter space. These two facts indicate that γ is highly responsible for the no identifiability of the model and that the possibility of fixing the parameter ϕ_1 instead of d for the estimation of the model should be also considered. With respect to the potential problem with γ, one might interpret this fact saying that the likelihood function is not able of extracting information about the parameter from the observed data.

Nieto (2002) has conducted more simulations with univariate and bivariate models finding the same results as above, that is, the coincident-index

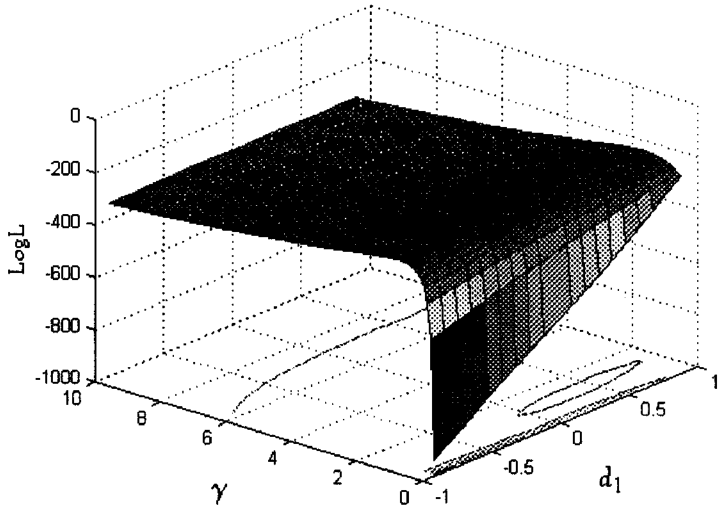

Figure 4: Likelihood surface for the second model with ϕ fixed.

statistical model is not identifiable, where γ becomes a troublesome parameter in this model. In the Appendix I include likelihood surfaces for a bivariate model that illustrate even more this point. In general, and thinking in the practice, I suggest fixing the d parameters and restricting the γ's to be nonnegative. However, for future research, another alternatives for fixing some model parameters must be considered, as indicated for the even partial simulation study of Nieto (2002).

In practice, I recommend to use the following strategy:

STAGE 1. Find reasonable estimates of the ds autoregressive parameters, via the fitting of a regression model with AR errors for each univariate equation in expression (1), where one takes an appropriate preliminary estimate of the process $\{C_t\}$ as the explanatory variable. Models adequacy may be checked with AIC/BIC information criteria or usual residuals-based statistical tests.

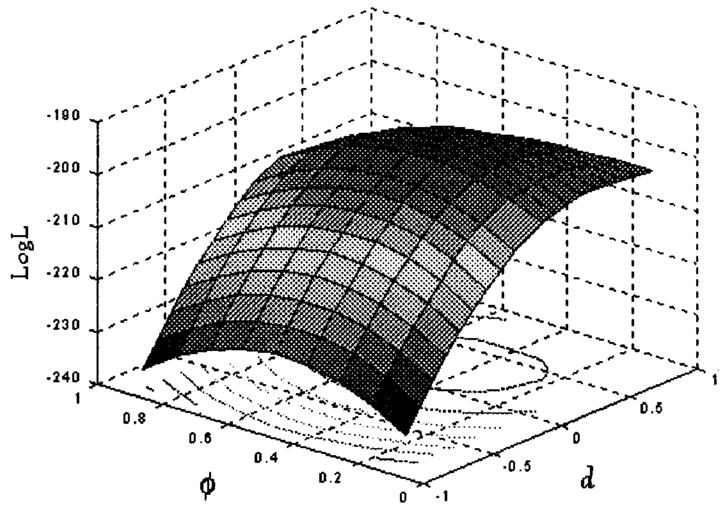

Figure 5: Likelihood surface for the second model with γ fixed.

As the common autoregressive order for the u variables, one can take the maximum of the marginal ones.

STAGE 2. In the estimation routine of the whole model, fix the d's parameters at the found values in STAGE 1 and restrict the γ's to be nonnegative.

The previous findings about the model identifiability and the proposed practical strategy are illustrated in a forthcoming paper about reestimating a Colombian coincident index.

4 Conclusions

In this theoretical report, I have found that the state space model used by Stock and Watson (1991), Kim and Nelson (1999), and Nieto and Melo (2001) for designing a coincident index in levels for the so-called state of the economy, is not identifiable. The latent process weights are highly responsible for this identification problem. In order to obtain the model identifiability property, I have sketched a practical approach in which the autoregressive parameters of the intrinsic processes are estimated previously to the likelihood function maximization and the latent process coefficients are restricted to be nonnegative.

A possible identification problem due to the simultaneous estimation of the autoregressive parameters of the latent process and the weights of it in the coincident equations should be investigated in the future. Simulation of the models considered here indicate that whenever the latent process weights are involved in the joint likelihood function, its surface tends to have large flat regions, which makes very difficult the likelihood-function maximization procedure for the model in levels. This simulation-based observation and the empirical fact of no convergence of the maximization routine in a forthcoming real application, favor the use of a transformed model as was done by Nieto and Melo (2001) and Melo *et al.* (2001).

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APPENDIX

In this appendix I present the likelihood surfaces with their respective contours corresponding to a bivariate model, i.e. $n = 2$, in which $\beta_t = \mathbf{0}$, $(\gamma_1, \gamma_2) = (1, 0.5)$, $(d_1, d_2) = (0.7, 0.4)$ and $(\sigma_1^2, \sigma_2^2) = (1, 0.5)$; $p = 0$ and $\delta = 0$. The likelihood function is $l(\gamma_1, \gamma_2, d_1, d_2)$ and there are 6 (the number of combinations of 2 elements among 4) possible surfaces to be analyzed. In Figures 6-11 I show the graphical results where one can see once more, the same situation about so many local maxima in the likelihood function or an absolute maximum that could be very hard to detect because of an almost flat hypersurface in its neighborhood, when the parameters γ are involved.

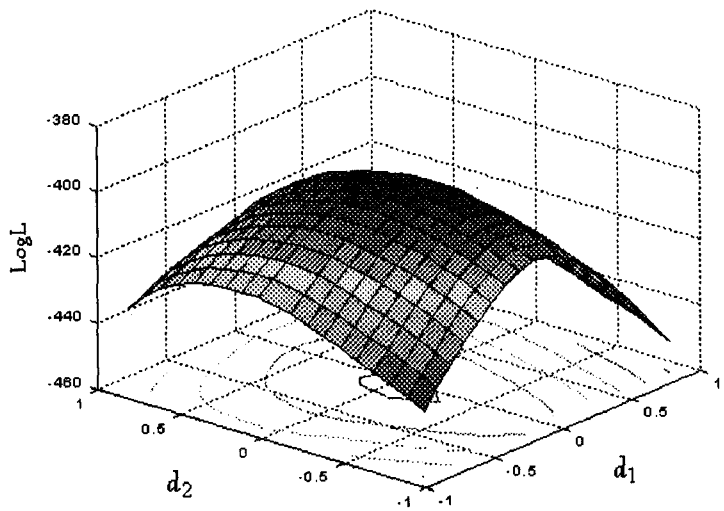

Figure 6: Likelihood surface for the bivariate model with fixed γ's.

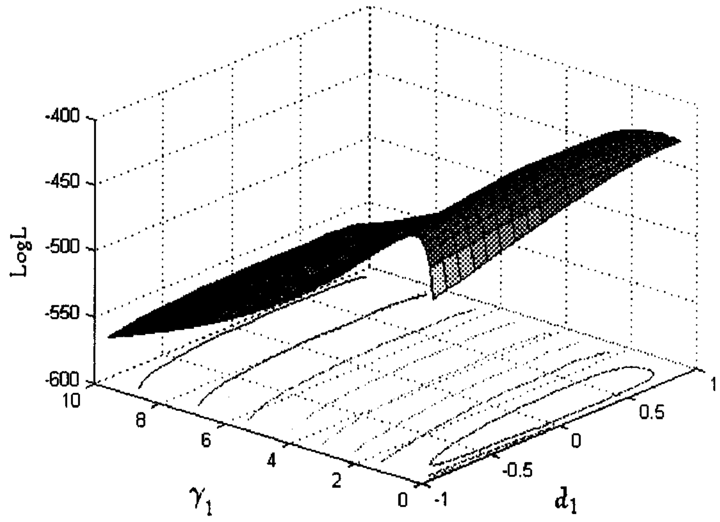

Figure 7: The surface of the bivariate model when γ_2 and d_2 are fixed.

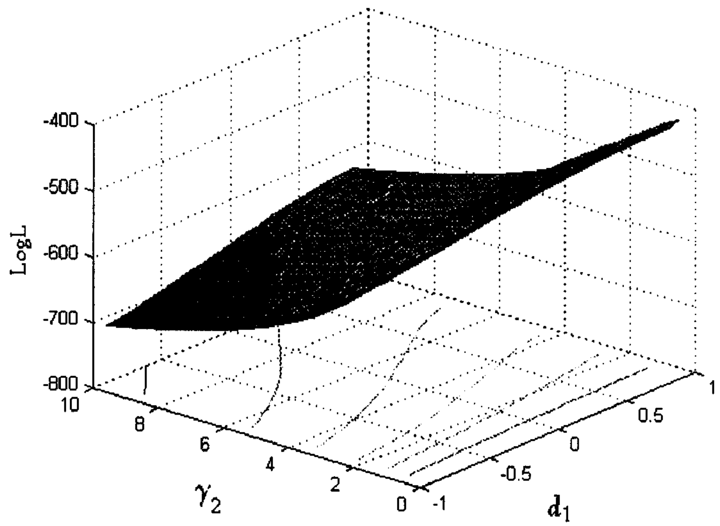

Figure 8: Likelihood surface when γ_1 and d_2 are fixed.

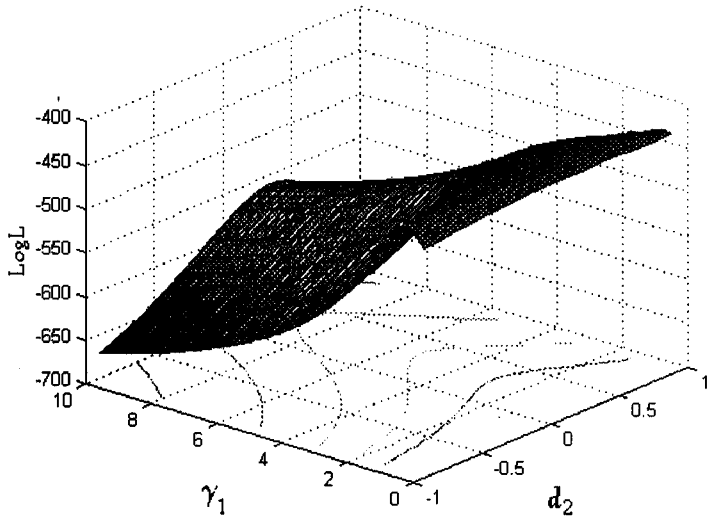

Figure 9: Likelihood surface for the bivariate model when γ_2 and d_1 are fixed.

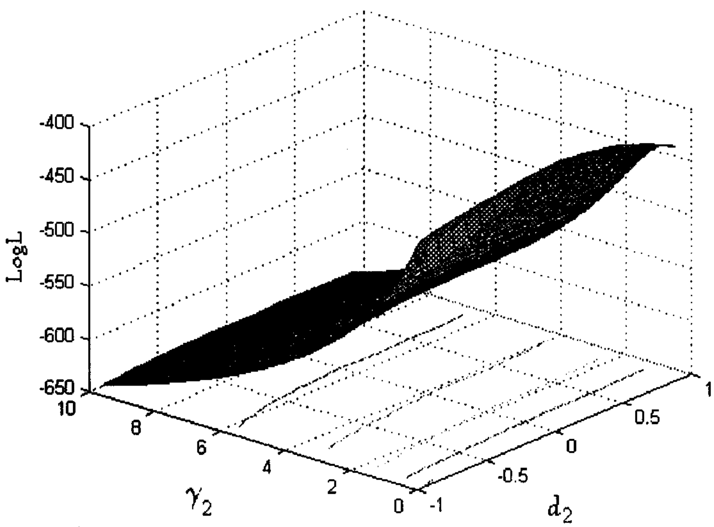

Figure 10: Likelihood surface when γ_1 and d_1 are fixed.

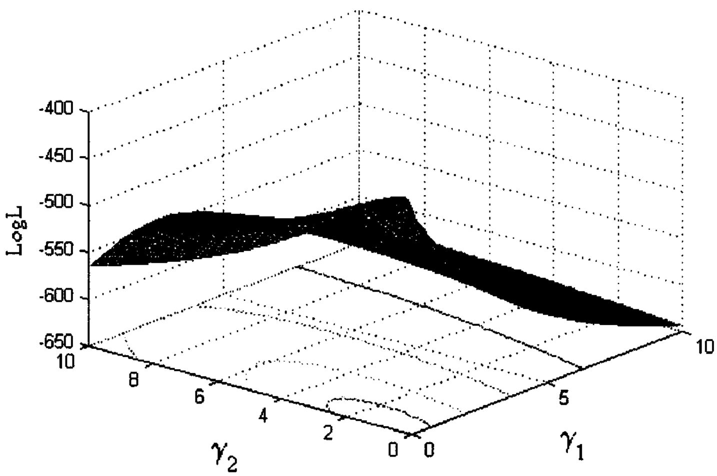

Figure 11: The surface for the bivariate model when d_1 and d_2 are fixed.