Efficient Portfolio Optimization in the Wealth Creation and Maximum Drawdown Space

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Abstract

First developed by Markowitz (1952), the *mean-variance* framework is the most widespread theoretical approximation to the portfolio problem. Nevertheless, successful application in the investment community has been limited. Assumptions such as normality of returns and a static correlation matrix could partially account for this.

To overcome some of the limitations of the *mean-variance* framework, mainly the choice of the risk metric and the inconvenience of using an estimated correlation matrix typical of tranquil or euphoria periods, this paper proposes an alternative risk measure: the *maximum drawdown* (*MDD*), and combines it with a *wealth creation* measure to define a new portfolio optimization space.

Like other market practitioners' measures, *MDD* lacks of a complete and solid theoretical foundation. In an effort to contribute to its theoretical foundation, this paper uses common sense and financial intuition to introduce such measure, followed by a review of its technical advantages and *coherence* for risk management.

Finally, an application of a *MDD* risk metric based portfolio optimization model is presented. The main findings indicate this proposal may effectively help overcome some of the traditional *mean-variance* shortcomings and provide some useful tools for portfolio optimization in practice.

For long-term performance driven portfolios, such as pension funds, this approach may yield interesting results because it focuses on *wealth creation* over the long run.

Keywords: Portfolio Optimization, Asset Allocation, Downside Risk, Maximum Drawdown, *mean-variance* Criteria, Diversification.

JEL Classification: G11, G23, G32, D81.

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1. Introduction

It is widely known that the Markowitz formulation of the portfolio optimization problem, based on maximizing expected return and minimizing risk, is the main pillar of the portfolio management theoretical foundations. Nevertheless, its limited impact in investment management practice is also widely recognized[1].

When analyzing the portfolio optimization problem for Colombian pension funds[2], the authors were confronted with the typical shortcomings of the Markowitz framework and faced some others not commonly discussed by the literature.

This paper presents an intuitive and convenient, theoretically robust, approach to reformulating the portfolio optimization problem. The latter mainly consists of a change in the solution space both for the metrics for risk - from standard deviation or variance to a market practitioners' measure known as *maximum drawdown*- and return – with the use of cumulative returns or end of period wealth-, as well as the optimization mechanics by using the actual time series instead of the estimated moments.

The paper is organized as follows: the first chapter presents a brief examination of the traditional framework for portfolio optimization. The second summarizes selected recognized problems of this framework, focusing on those related to the choice of risk measure. Afterwards, based on some desirable properties, both theoretical and practical, the change of risk measure from dispersion to *maximum drawdown* (*MDD*) is justified. Next, a *MDD* based case of portfolio optimization is presented. Finally, some remarks are discussed.

2. The Markowitz framework for portfolio optimization

The main contribution of the Markowitz (1952) formulation consists of recognizing that rational behavior of investors is better represented by the rule of considering expected return as a desirable and variance of return as undesirable, instead of the short-sighted hypothesis of merely maximizing discounted returns prevailing back then. This is known as the *mean-variance criteria* (*MVC*), which states that when an investor faces two portfolios, *A* and *B*, he will prefer portfolio *A* to *B* when:

$$E(r_A) \geq E(r_B)$$

and

$$\sigma^2(r_A) \leq \sigma^2(r_B)$$

where

[1] Some of the literature on this subject and on the failure of academic models in practice is He G. and Litterman R. (1999), Pedersen C. S *et al.* (2003), Pézier J. (2007), Chhabra A. B. (2005), Bhansali V. (2005), among others.
[2] Reveiz A. and León C. (2008) and Reveiz A., *et al.* (2008).

$E(r_i)$ Asset's *i* expected return

$\sigma^2(r_i)$ Asset's *i* variance

When formalizing portfolio's return-risk framework, Markowitz identified the benefit of diversification as the milestone of modern portfolio theory. Despite being and old and widely used concept, diversification's first mathematical formalization was provided by Markowitz, who was also the first to recognize numerically how diversification can reduce risk for a given level of expected return[3]. For the case of N assets, portfolio's expected return and variance are calculated as follows:

$$E(r_p) = W^T E(r)$$

[1]

where

$E(r_p)$ Portfolio's expected return

$W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$ Column vector of assets' weights

$E(r) = \begin{bmatrix} E(r_1) \\ E(r_2) \\ \vdots \\ E(r_N) \end{bmatrix}$ Column vector of assets' expected returns

and

$$\sigma^2(r_p) = W^T \Omega W = \begin{bmatrix} w_1 & w_2 & \cdots & w_N \end{bmatrix} \begin{bmatrix} Var(j,j) & Cov(j,k) & \cdots & Cov(j,N) \\ Cov(k,j) & Var(k,k) & & \vdots \\ \vdots & & \ddots & \vdots \\ Cov(N,j) & \cdots & \cdots & Var(N,N) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

[2]

where

$\sigma^2(r_p)$ Portfolio's variance

$\Omega = \begin{bmatrix} Var(j,j) & Cov(j,k) & \cdots & Cov(j,N) \\ Cov(k,j) & Var(k,k) & & \vdots \\ \vdots & & \ddots & \vdots \\ Cov(N,j) & \cdots & \cdots & Var(N,N) \end{bmatrix}$ Covariance matrix

[3] Rubinstein M. (2002).

Using the expected return-standard deviation space and introducing the quadratic approximation to portfolio risk presented above, Markowitz was able to demonstrate the existence of a set of efficient combinations of expected return and risk which is commonly known as the *Efficient Frontier* (*EF*). Generally, for each point on the *EF*, the optimization procedure is carried out with the following quadratic program[4]:

$$\min \sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} = W^T \Omega W$$

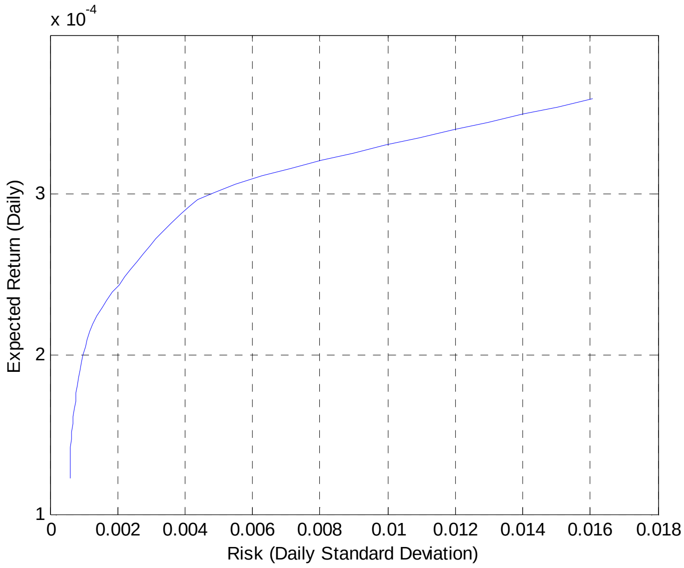
[3]

s.t. $$\sum_{i=1}^{N} w_i = 1$$

$$\sum_{i=1}^{N} w_i E(r_i) = E(r_p)$$

The *EF* is a plot resulting from the optimization process above, in which for each level of expected return the minimum expected risk is attained, where all portfolios below the minimum-variance portfolio are discarded.

Figure No.1
Markowitz's *Efficient Frontier*



Source: authors' calculations

Many other contributions to portfolio theory have been made since Markowitz seminal work, but most of portfolio theory still relies on its foundations. Surprisingly, despite the fact that this theoretical framework is extensively known and has survived the test of time, it's not widely applied to practical asset allocation. Some of the reasons why the optimization problem within the Markowitz framework is not practical are: the portfolio

[4] Cuthberson K. and Nitzsche D. (2004). Some other constrains as the non negativity of weights (no short-sale restriction) are commonly included.

weights' sensitivity to input changes; extreme and non-intuitive portfolio weights[5]; historical estimation of risk and correlation has proven to fail constantly in practice[6], among others.

3. Risk Measurement and the Portfolio Problem

Dealing with some of the recognized shortcomings of modern portfolio theory for pension fund's portfolio management[7], the authors faced some others related with i) the risk measurement and ii) the way investors would optimize their portfolio taking into account the risk measure.

The first shortcoming relates to the choice of a risk measure. Contrary to risk, return is a rather clear concept and its calculation for an asset or a portfolio is straightforward. The ordinary statistical measure of risk is volatility, a metric of dispersion which measures the size of a typical observation's departure from its expected value. Litterman (2003) recognize two main sources of weakness of volatility as metric for risk: i) only in special cases, such as normally or Gaussian distributed returns, volatility alone can provide enough information to measure the likelihood of most events of interest, namely extreme events, and ii) volatility is a measure of risk that does not differentiate upside risk from downside risk, a rather important issue when considering non symmetric distributions.

Beyond Litterman's arguments, Taleb (2004, 2007) discusses the origins of the usage of Gaussian distributions and volatility in finance. He concludes that people in finance just borrowed a technique from disciplines which don't have problems eliminating extreme values from their samples, such as education and medicine. Taleb argues that due to the fact that concepts such as standard deviation and correlation do not exist outside the Gaussian world, and because in such a world the odds of a deviation decline exponentially as departing from the mean, relying on Gaussian distributions when dealing with aggregates where magnitudes do matter, such as portfolio management, implies ignoring unpredictable large deviations, which, cumulatively, show a dramatic impact on wealth.

This risk metric issue is not new and was first mentioned by Markowitz (1952). Some authors[8] have tried to solve the weakness of volatility as measure of risk and found that using metrics such as *Value at Risk* (*VaR*) and *Expected Shortfall* generate *Efficient Frontier*s which are subsets of the *mean-variance* frontier if, and only if, the normality, or at least ellipticality, assumption holds. Nevertheless, the allocations obtained by these authors are similar to those of the *mean-variance* frontier and the normality of returns' and the equal treatment to upside and downside risk usually remains[9].

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[5] He G. and Litterman R. (1999).
[6] Bhansali V. and Wise M. (2001) and Greenspan A. (2008).
[7] Previous works from the authors concerning pension funds and their portfolio optimization problem for the Colombian capital market are Reveiz and León (2008) and Reveiz, *et al.* (2008).
[8] De Giorgi E. (2002) and Hurlimann W. (2002). For a review of other authors and alternative measures of risk please refer to Pedersen C. S, *et al.* (2003).
[9] The latter shortcoming, the equal treatment given by VaR and Expected Shortfall to upside and downside risk, can be surmounted calculating standard deviation or variance by replacing the upside

When analyzing the impact of implementing a multifund scheme for Colombian pension fund's portfolio management, the authors confronted concerns the way final investors, namely the future pensioners or the "*ordinary man*", behave and decide under uncertainty. At first it is reasonable to assume that any individual has monotonic preferences and will be non-satiated in consumption (will prefer more rather than less of a good); according to this, in what is known as *dominance*, an investor will always prefer the investment that pays as much in all states of nature, and strictly more in at least one state[10].

But, if an investor faces alternative investment opportunities not covered by the *dominance* concept, namely if there is no investment that pays as much in all states of the nature and strictly more in at least one, the decision becomes less clear and the *mean-variance dominance* concept emerges. According to the *mean-variance dominance*, an individual will characterize the investment opportunities by their first two moments (mean and variance) and decide accordingly: for investments of equal magnitude, of the same expected return (mean), choose the one with the lowest risk (variance); for investments of the same magnitude, with the same risk (variance), choose the one with the greatest expected return (mean).

Even though the *mean-variance criteria* or *mean-variance dominance* is the milestone of the modern portfolio theory, the variance or standard deviation choice as a metric for risk for an individual is far from being practical and meaningful for what we can call an "*ordinary man*". Not only because traditional calculations of variance relies on normality of returns and gives equal treatment to upward and downward risk, but because it's subject to estimation errors and escapes from the knowledge and understanding of the common individual, variance may not be the best measure of risk.

Despite not receiving much attention, Roy (1952) developed an alternate mathematical foundation to the optimization problem in parallel to Markowitz. Roy, concerned with the "*ordinary man*" behavior under uncertainty, deviates from Markowitz framework when defining the appropriate metric for risk: he develops the "*safety first*" concept.

The "*safety first*" concept tries to handle two main observations made by Roy (1952):

a) *The ordinary man has to consider the possible outcomes of a given course of action on one occasion only; the average (or expected) outcome, if this conduct were repeated a large number of times under similar conditions, is irrelevant.*
b) *Is it reasonable that real people have, or consider themselves to have, a precise knowledge of all possible outcomes of a given line of action, together with their respective probabilities or potential surprise?* [11]

Roy, concerned with these observations, recognizes that given observation a) disasters do exist and are the investors' most important source of concern, and because of b) investors generally suffer from limited knowledge. Consequently he develops the principle of "*safety first*", which asserts that is reasonable, and probable in practice, that

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values by zero as noted by Dowd K. (2005). Other way to deal with this shortcoming is the use of lower partial moments as noted by Pedersen C. S, *et al.* (2003).

[10] Danthine J-P. *et al.* (2002).

[11] His view was closer to Popper's (1991) propensities and the Complexity discipline (See Reveiz., 2008).

an individual, given his lack of knowledge, will simply reduce the chance of a disaster from happening.

When facing the portfolio optimization problem, Roy asks: "If the prices of all the other assets fell to the critical price, what is the best (linear) estimate of price of the asset under examination?" Afterwards, he concludes that the best structure of assets is the one which keeps the chance of disasters happening as small as possible at the end of a given period of time, alternatively supporting in this way the portfolio diversification principle[12].

Despite being somewhat obvious that disasters should be avoided, recent market developments show that prevailing risk models have performed poorly, mainly because they are not designed to really perform under stress.

As Greenspan (2008) points out, state-of-the-art statistical models perform poorly because the underlying data is generally drawn from both euphoria and fear periods, which show very different dynamics, namely length and magnitude. Because contraction phases are far shorter[13] and far more abrupt, prevailing risk model's correlation benefits -based on average co movements-, evident during euphoria or calm periods, collapse as all the prices fall together, rendering the models ineffective. This argument is shared by Bhansali (2005) and Zimmermann *et al.* (2003).

Consequently, Bhansali (2005) and Zimmermann *et al.* (2003) emphasize that since most *Value at Risk* (*VaR*) and shortfall models are based on historically estimated covariance matrices, they are notorious for failing when most needed. Both agree that the difference of volatility and correlations between up and down market environments implies that the risk reduction potential of diversification is limited in down markets, thus making such models exhibit a downward bias and unable to foresee stress-type events. Furthermore, Bhansali (2005) highlights that most models lean very heavily on the notion of stability by assuming stable distributions for the dynamics of prices, which clearly ignores the impact of structural breaks and market discontinuities. Taleb (2007), as already mentioned, blames reliance on Gaussian distributions for the poor performance of finance when dealing with unpredictable large deviations, sharp jumps or discontinuities.

Thus, it can be pointed out that the lack of knowledge of the "*ordinary man*" is perhaps not exclusive to uninformed individuals but shared even by experienced market participants.

[12] Roy D. (1952) develops mathematically an approximation to the benefit diversification. His work with the disaster measure includes its usage as a threshold for investment decisions, which is not the goal of this paper.

[13] Greenspan A. (2008) points out that over the past half-century the American economy was in contraction only one-seventh of the time. Based on IMF's International Financial Statistics (IFS) annual industrial production index, the authors estimated that in the 1948-2006 period the 22 industrial countries were in contraction (negative growth) between one-fourth and one-fifth of the time, meanwhile the United States alone was nearly one-sixth of the time. For Colombia, also based on the IFS, the annual GDP volume index in the 1968-2006 period was in contraction (negative growth) only in one-thirty-eighth of the time.

Thus, the inclusion in portfolio construction of a risk measure that better deals with disasters and accounts for our lack of knowledge, the existence of novelty, and the animal spirits and irrational behavior that govern financial markets, is desirable.

4. Disasters and *Maximum Drawdown* as Measure of Risk

In several topics in finance practitioners tend to twist the theoretical commonly accepted model to make it useful when facing the realism of the market practice[14] or just develop measures that pay attention to what they care most. This is the case for risk, where many industry measures are used in spite of somewhat weak theoretical foundations.

In line with Roy's (1952) worries about the convenience of minimizing the occurrence of disasters, some "*safety first*"-type measures are being used by money management professionals. One of these measures is the m*aximum drawdown (MDD)*.

Defined as the maximum sustained percentage decline (peak to trough) which has occurred in an investment (individual asset or portfolio) within a period, the *MDD* provides an intuitive and easy to understand measure of the loss arising from potential extreme events.

The *MDD* calculation is not available in a closed-form formula and should be calculated recursively[15]. When calculating *MDD* for period [0,T], let V_T be the end dollar value of the series and V_{max} the maximum dollar value of the series in the [0,T-1] period, given the prior calculation of *MDD* for [0,T-1],

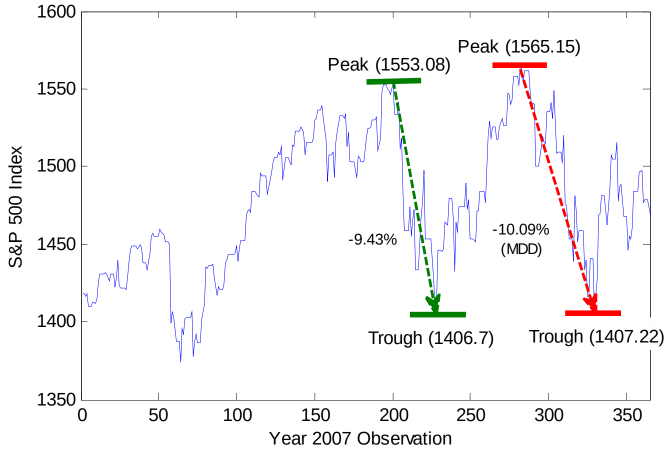
$$MDD_{[0,T]} = \min\left(\frac{V_T - V_{\max}}{V_{\max}}, MDD_{[0,T-1]} \right)$$

[4]

The figure below presents the *MDD* concept. Using 2007 data for S&P 500 index, *MDD* risk measure is calculated. For the sake of comparison the two sharpest declines are presented. The sharpest decline of those two (red arrow) corresponds to the drop from observation (day) 282th to 330th, a 10.09% drop, which is the *MDD* for S&P 500 index during 2007.

[14] Perhaps the most famous and studied case is practitioners' adjustments to the Black & Scholes option pricing model. Practitioners, in order to make the pricing model useful, violate the theoretical assumption of constant volatility and just plug the volatility surface to approximate market observed prices.
[15] Lohre H. *et al*. (2007).

Figure No.2
S&P 500 Maximum Drawdown



Source: authors' calculations

Following Roy (1952), the *MDD* measure provides valuable information because for the investor the average outcome is simply irrelevant, being really concerned about the outcome of his decision on that occasion when a disaster may seriously erode his wealth. Any rational individual, when confronted with two assets with the same return, will prefer the one with the lowest *MDD*, as is the case with variance in the *mean-variance criteria*.

When facing the pension fund problem in Colombia, Reveiz and León (2008) had to deal with very long term investment decisions, with investors quite sensitive to the long run wealth destruction due to sharp mark to market driven loses[16]. Similarly, Magdon-Ismail and Atiya (2004) recognize that most trading desks are interested in long-term performance, that is, systems that can survive over the long run, with superior return and small *drawdowns*; they state that a reasonably low *MDD* is in fact critical to the success of any fund.

As some financial market practitioners, the authors found that the *MDD* provides a useful yet intuitive and sound market risk metric, which deal with risk in ways variance cannot offer. Some major advantages are: i) *MDD* only comprises downward risk, which is a desirable property when considering the issue about the period (euphoria or fear) from which the model inputs are drawn; ii) because it corresponds to a proxy of the magnitude and length of disaster, *MDD* gives a better picture of how the market discontinuities and irrational behavior may look like; iii) since it relies directly on historical returns, *MDD* conveniently avoids normality –or any distributional-assumptions and estimation errors.

Despite these serious advantages, *MDD* is not a widely discussed topic in financial theory. In an effort to contribute to its theoretical foundation, following Artzner *et al.* (1998), who developed a formal theory of financial risk, we now evaluate if *MDD* may be generally regarded as a convenient and sound risk measure.

[16] The problem is compounded by the fact that contributions are not made constantly, i.e. the probability of an affiliated to contribute in a given month can be lower than 40%.

In an attempt to conceptualize what a risk metric should be Artzner *et al.* (1998) postulated a set of axioms which ensure that a risk measure is -as they call it- *coherent*; that is, it's a risk measure which is to be used to effectively regulate or manage risks.

Based on Dowd (2005) and Cheng *et al*. (2004), let X and Y represent any two random variables, may them be the changes in values of an investment, and let $\rho(.)$ be a measure of risk, which represents the minimum extra cash that has to be added to the risky position in order to make it acceptable. The measure of risk $\rho(.)$ is *coherent* if it satisfies the following properties:

a) *Monotonicity*: $Y \geq X \Rightarrow \rho(Y) \leq \rho(X)$. This means that if two random variables representing dollar changes in the values of investments, X and Y, are such that $Y \geq X$, then their risk measures have to satisfy $\rho(Y) \leq \rho(X)$. In other words, because the dollar change in value of investment Y is always higher than X, the latter should be compensated so it is acceptable to hold it, then it can be identified as more risky[17].

b) *Subadditivity*: $\rho(X+Y) \leq \rho(X)+\rho(Y)$. This means that the measure of risk of a portfolio composed by X and Y should always be equal or lower than the sum of the risk of X and Y alone. This property, the single most important and desirable of them all, reflects that any reasonable risk measure should aggregate individual risks in such a way that there is some reduction, or at least not an increment when compared to the simple sum of individual risks; otherwise, firms or investors would be tempted to break up their accounts or investments in order to reduce risk.

c) *Positive homogeneity*: $\rho(hX) = h\rho(X) \quad \text{for any } h > 0$. This means that the risk of a position is proportional to its scale or size, which makes sense when the positions are liquid; if the positions or instruments are not liquid enough there may be the case for $\rho(hX) \geq h\rho(X) \quad \text{for } h > 0$, just because to sell a large position may confront liquidity risk.

d) *Translation invariance*: $\rho(X+n) = \rho(X) - n \quad \text{for some certain amount } n$. This means that the addition of a sure amount reduces the cash needed to make the position acceptable. It is of great importance because if the sure amount n is equal to $\rho(X)$, then $\rho(X+\rho(X)) = \rho(X) - \rho(X) = 0$, which is a neutral position.

The following items provide the intuition behind the performance of the *MDD* risk measure for each property:

a) Concerning *monotonicity*, *MDD* offers a measure which assures that the investment with lowest performance should be compensated in order to make it acceptable to hold. If *MDD* is calculated for two random variables representing

[17] For the sake of comprehension of the *monotonicity* property the reader should avoid thinking of risk using conventional dispersion approach. Dispersion-type measures of risk, such as standard deviation or variance, don't comply with this property; dispersion measures don't differentiate between the signs of the random variables, thus they would only reveal the magnitude of the changes in value of an investment X or Y, not their direction.

changes in dollar values of an investment, *X* and *Y*, and the dollar value of the random variable *Y* is always higher than *X*, then *X* happens to be riskier. It is worth mentioning that a special case for this property can be found: because *MDD* is zero when applied to a strictly increasing price time-series, when dollar value changes of *X* and *Y* are strictly positive then $\rho(Y) = \rho(X) = 0$.

b) *Subadditivity* is guaranteed given the computation of a portfolio's *MDD* results from a linear combination of the returns of the individual assets. Moreover, given that not all individual asset's extreme results always happen at the same time, there may be a diversification gain (reduction in aggregated risk) because individual disasters tend to be averaged out; the portfolio's *MDD* diversification effect does not rely on the estimation of a correlation matrix, then its free of estimation errors. Even if there is a major catastrophe in which strictly all assets exhibit major adverse movements at strictly the same time, the *MDD* will equal the weighted average of individual *MDD*'s, without any diversification gain. Then, the $\rho(X+Y) \leq \rho(X) + \rho(Y)$ property will always hold.

c) Regarding *positive homogeneity*, due to the fact that *MDD* is defined as the maximum sustained percentage decline (peak to trough) which has occurred in an investment within a period, scaling the value of the dollar value-random variable is simply vain: the *MDD* will be the same. For the *MDD* to comply the *positive homogeneity* it suffices to adjust the way this measure is presented: if we convert the *MDD*, which is by construction a percentage, into an absolute monetary value by simply multiplying it by the size of the nominal position, this property will always hold; this is equivalent to creating a dollar-*MDD* measure.[18]

d) Finally, the *translation invariance* is complied with a similar presentation adjustment of the *MDD* measure. If converted into an absolute monetary value by multiplying *MDD* by the size of the nominal position, which is equivalent to creating a dollar-*MDD* measure, this property will hold. These two conversions, for properties c) and d), are by no means a twist of the properties: it just takes into account the fact that the risk measure $\rho(.)$ represents the minimum extra cash that has to be added to the risky position in order to make it acceptable, not a percentage value as is the case with *MDD*.

According to the intuition presented, and based on some empirical tests when deemed necessary[19], the authors conclude that besides being an intuitive and sound risk metric used by market practitioners, *MDD* can be regarded as a *coherent* risk measure in the Artzner *et al.* (1998) sense, thus useful to effectively regulate or manage risks.

[18] Because is outside the scope of this paper we disregard the liquidity issue that may cause the violation of the *positive homogeneity* property.

[19] For properties b), c) and d) the authors carried out several empirical –random- tests that can be provided by request.

5. The Portfolio Optimization Problem under the Maximum Drawdown risk measure

Given the practical advantages and the *coherence* of *MDD* as a measure of risk, it's tempting to use the *Efficient Frontier* (*EF*) framework replacing the dispersion (variance or standard deviation) by *MDD*.

In Markowitz's *mean-variance criteria* (*MVC*) the *EF* results from an optimization which attains the lowest possible portfolio dispersion for a given portfolio expected return. The portfolio optimization proposed by this paper differs from the *MVC*'s not only in the risk measure: looking again to market practitioners' alternative measures, the approximation to expected return via the average of past returns is replaced by the total past effective return, which is simply the wealth created by a portfolio over the period studied.

Using the total return as measure of expected return and *MDD* as measure of risk we find the *Calmar Ratio* (*CR*), which is a measure used by some portfolio managers despite no explicit theoretical support exists[20]:

$$CR_{(i,t)} = \frac{TR_{(i,t)}}{MDD_{(i,t)}}, \text{ where}$$

[5]

$TR_{(i,t)}$	Asset's *i* Total Return over the period *t*
$MDD_{(i,t)}$	Asset's *i MDD* over the period *t*

CR is a risk-adjusted performance measure which presents the trade-off between *wealth creation* and risk, with the latter considered as a measure of disaster.

Then, when tackling the portfolio problem in the hereby proposed modified return-risk framework, the *EF* results from an optimization which attains the lowest possible *MDD* for a given portfolio *wealth creation* level, once all the portfolios below the minimum *MDD* portfolio are discarded. For each point on the frontier, the optimization procedure would be carried out as follows:

$$\min MDD_p = MDD(A \cdot W)$$

[6]

$$\text{s.t.} \quad \sum_{i=1}^{N} w_i = 1$$

[20] Pedersen C. S. *et al.* (2003) and Magdon-Ismail M. and Atiya A. (2004).

$$\sum_{i=1}^{N} w_i TR_i = TR \cdot W^T = TR_p$$

$$w_i \geq 0$$

where

TR_i	Asset's *i* Total Return
TR_p	Portfolio's Total Return
$TR = \begin{bmatrix} TR_1 \\ TR_2 \\ \vdots \\ TR_N \end{bmatrix}$	Column vector of assets' Total Returns
$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{K,1} & \cdots & \cdots & a_{K,N} \end{bmatrix}$	Matrix (time series) of asset prices (*K observations, N assets)*
$W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$	Column vector of assets' weights

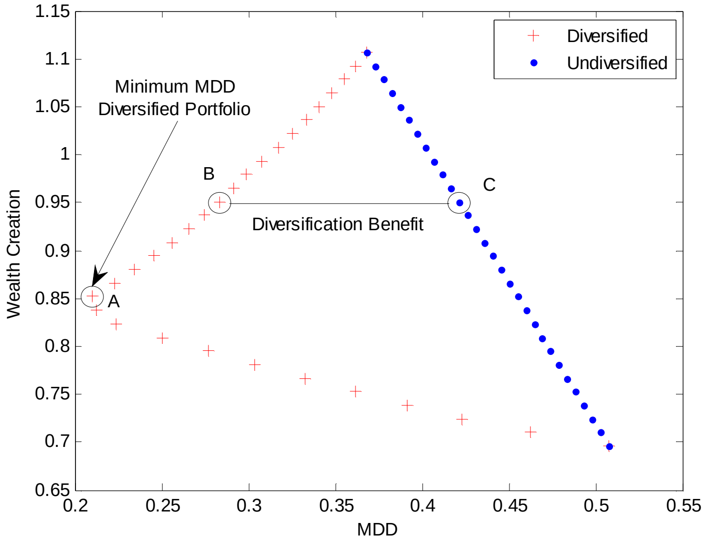
Due to the fact that the s*ubadditivity* property holds and the fact that there exists a diversification gain when combining assets, we could expect to find an *EF* to some extent similar in shape to Markowitz's, with great advantages: i) no correlation matrix has to be estimated, the benefit of diversification corresponds to the realized risk reduction due to the combination of assets; ii) the resulting diversification benefit corresponds to the risk reduction when it matters most: in the middle of disaster; iii) the risk metric used is free of the dispersion-type risk measures' shortcomings; iv) no distribution is assumed.

Using a set of daily prices for 18 assets or risk factors from February 1990 to December 2007, comprising commodities, equity indexes, sovereign fixed income indexes of four major industrialized economies, US mortgages and corporates[21], we construct the

[21] The chosen asset classes and individual assets or risk factors are described next: Commodities (WTI oil, wheat, cotton, platinum and gold), equity indexes (Nasdaq, MSCI, MSCI- EM), sovereign fixed income indexes (US Treasuries 5-10years, US Treasuries 10+years, Germany 5-10years, Germany 10+years, United Kingdom 5-10years, United Kingdom 10+years, Japan 5-10years, Japan 10+years), US Mortgages and US AAA Corporates. They all are daily series, and were obtained from Bloomberg, where the sovereign fixed income indexes, US Mortgages and AAA Corporates correspond to Merrill Lynch indexes. MSCI and MSCI-EM correspond to the MSCI All Country World Index and MSCI Emerging Markets Index, where the former is a 49 markets total market capitalization of all securities index, while the latter only comprises 27 emerging markets.

complete spectrum of w*ealth creation-MDD* attainable by different combinations of assets or risk factors. For instructive purposes we begin by combining two risk factors, where each mark corresponds to one of the 30 portfolios constructed for each frontier, diversified and undiversified:

Figure No.3
Wealth Creation-MDD's *Frontier*
(2 assets: gold and MSCI)



Source: authors' calculations

The red-crosses line correspond to the lowest possible *MDD* for a given portfolio w*ealth creation* level, with portfolio's *MDD* calculated as shown in equation *[6]*, thus the diversified frontier. The blue-dots line corresponds to the lowest possible *MDD* for a given portfolio w*ealth creation* level, with portfolio's *MDD* simply calculated as the weighted average of each asset's *MDD*, which corresponds to a major catastrophe in which strictly all assets exhibit major adverse movements at strictly the same time, thus, the undiversified frontier.

The interpretation of a specific portfolio such as *B* could be as follows: a certain combination of gold and *MSCI* provides an investment opportunity which in the analyzed period (2000-2007) attained a total return or *wealth creation* of 95.1% and was exposed to a 28.3% *MDD*.

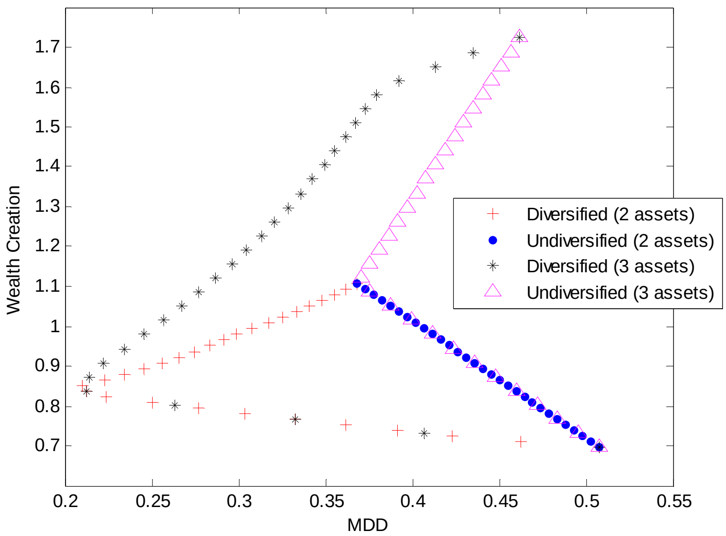
The *A* portfolio corresponds to the minimum *MDD* diversified portfolio, equivalent to Markowitz's minimum variance portfolio; portfolios with a *wealth creation* equal or above *A* would set up the *Efficient Frontier*. The horizontal difference between the diversified and undiversified frontiers for each level of wealth represents the benefit of diversification attained by the combination of assets within the portfolio; because the diversified frontier dominates the undiversified, the *subadditivity* property is once again verified.

Next we include one more asset, the MSCI-Emerging Markets (MSCI-EM) index, and analyze the results:

Figure No.4

Wealth Creation-MDD's *Frontier*

(3 assets: gold, MSCI and MSCI-EM)



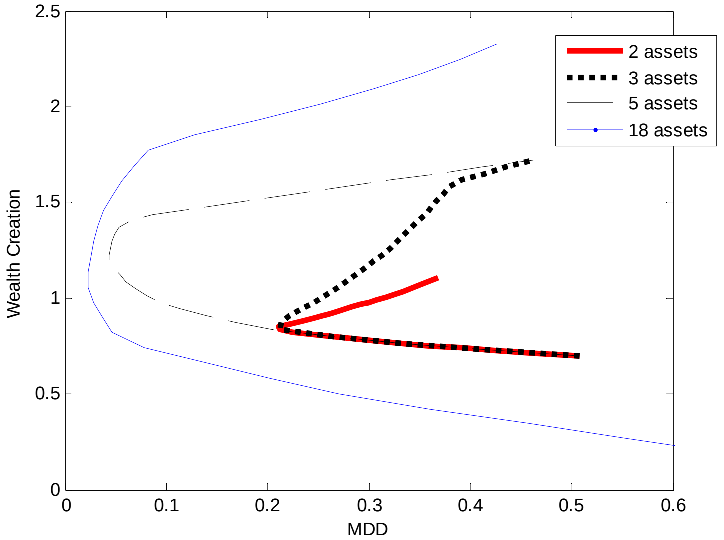
Source: authors' calculations

As with traditional Markowitz-type portfolio optimization, the addition of an extra asset expands the range of return and risk combinations. The 3-asset diversified frontier, represented by the black asterisks, dominates the 2-asset frontiers (red crosses). We now introduce two more frontiers, with 5 and 18 assets, but drop the undiversified frontiers for graphic ease.

Figure No.5

Wealth Creation-MDD's *Frontier*

(2, 3, 5 and 18 assets[22]*)*

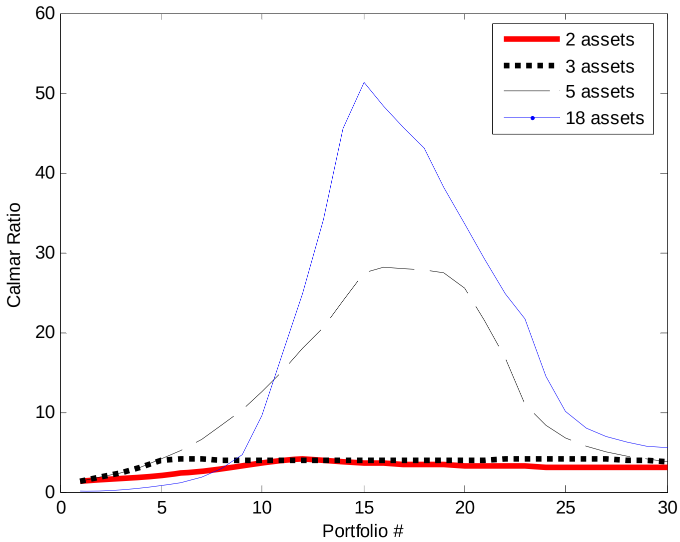


Source: authors' calculations

[22] The 2 assets frontier corresponds to the combination of gold and MSCI; the 3 assets frontier corresponds to gold, MSCI and MSCI-EM; the 5 assets frontier corresponds to gold, MSCI, MSCI-EM, US Treasury 5-10y and US AAA Corporates. The 18 assets frontier adds commodities, other equity indexes, other sovereign fixed income indexes and US mortgages.

The *Calmar Ratio* (*CR*), calculated as in *[5]*, is plotted in the following figure:

Figure No.6
Calmar Ratio
(2, 3, 5 and 18 assets)

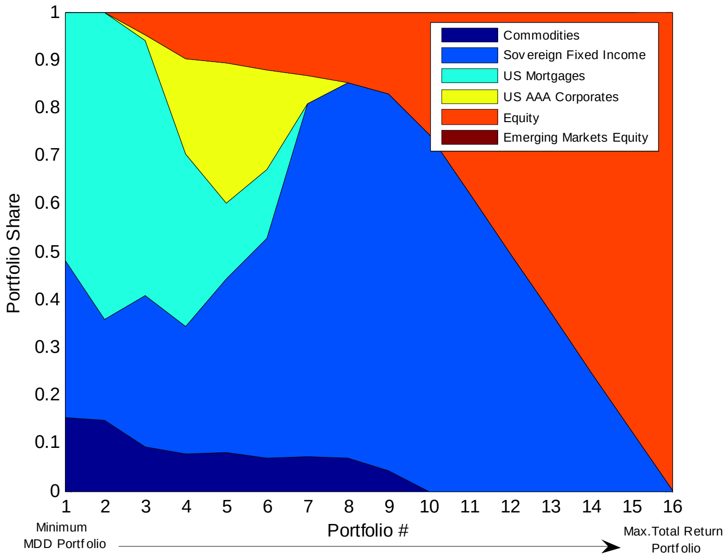


Source: authors' calculations

An investor seeking to maximize the *wealth creation* per unit of risk (*MDD*) would choose the highest *CR*. The 18-assets frontier achieves the highest *CR* levels, followed by the 5, 3 and 2-asstes frontiers respectively. These results show that the higher the number of assets or risk factors, the higher the *CR* attainable due to the diversification effect, which partly comes from disasters' averaging out as more risk factors are added to the portfolio.

Finally, after discarding the non-efficient portfolios, the 18-asset *EF* is composed of 16 portfolios. Figure No.7 presents the portfolio breakdown by asset class, where the first portfolio is the minimum MDD portfolio, and moving along the x-axis provides the composition of EF's riskier portfolios.

Figure No.7
Efficient Frontier Composition
(18 assets, by asset class)



6. Final remarks

Despite it has been widely recognized by practitioners the inconvenience of facing the portfolio problem with the standard-academic tools, and notwithstanding the evident poor performance of contemporary state-of-the-art statistical models when confronted with critical events, practitioners lack of practical models which approach risk in a meaningful and sound manner.

This document has presented a first attempt to what the authors expect to be a rather novel method for approaching the portfolio problem. The novelty comes from five directions: i) *MDD* risk metric deals with what risk models should be more concerned about: extreme adverse events; ii) *MDD* risk metric, because it does not rely on the assumption of full knowledge of the probability distribution of future outcomes, is more convenient and sound than traditional –dispersion- metrics; iii) when calculated as in *[6]*, portfolio's *MDD* benefits from combining assets, but it does without estimation errors and focused on the diversification that matters the most for an investor: in extreme adverse events; iv) despite being a practitioners' risk measure, authors' analysis and empirical tests have proven *MDD* has interesting properties which meet formal theoretical *coherence* criteria; v) w*ealth creation* and *MDD*, together provide a new space for portfolio optimization.

The authors recognize some critical issues which should be properly tackled. As with Markowitz and the majority of approaches to the portfolio problem, the length of the time series and its periodicity used is extremely relevant for the optimization result. Due to the fact that the purpose of a risk metric such as *MDD* is to address extreme market movements and discontinuities, the authors suggest using a considerable amount of information, which allows the model to find each asset's infrequent but critical wealth-destructive rare event and its marginal risk diversification impact when added to a portfolio; it is compulsory to use enough information to cover at least a complete business cycle, but is always advisable to cover more than one.

Other issue concerns the extreme event anticipating power of the model. Despite this model –or any model- is incapable of anticipating the origin, magnitude and length of the next extreme event to come, the authors find that using *MDD* provides the investor with a more realistic and sound risk measure, specially when compared to traditional state-of-the-art models based on Gaussian distributions and the estimation of volatility or correlation.

Finally, the main drawback of our proposal relates to the computational resources needed. The *mean-variance* and other distribution moment optimization procedures just require finding the asset's weights which achieve the minimum risk for each level of return, where portfolio's risk results from a closed-form formula such as standard deviation or variance -an easy task for any standard optimizer. Our proposal, based on the calculation of *MDD* as in *[6]*, does not rely on any moment estimation, and thus requires a more complex and time demanding optimization procedure[23] that finds the weights for each asset's time series -again, not the moments- which minimizes the *MDD*, a non-closed-form risk measure.

Although this work gives a preliminary insight on optimizing under the wealth and *MDD* space, further research is needed. Some of this research will come in the form of methodologies to compare ex-pot returns under constant and rebalancing scenarios with other asset allocation techniques, and proper *backtesting* methods for comparing the *mean-variance* space optimization with the one proposed. An application for Colombian pension system is also in the authors' agenda.

[23] Using Matlab® the computational time required to find the portfolio 30-portfolios frontier in the *wealth creation –MDD* space was about 598.6 seconds for the two asset case, while the time required for the *mean-variance* space was about 3.5 seconds; in order to make models comparable in precision and programming quality, we didn't use Matlab's ® Financial Toolbox portfolio optimization functions, but our own optimization code.

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[24] This document has no year of publication. However, according to SSRN (www.ssrn.org), it was first posted November 28th 2001.

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