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Portfolio Optimization and Long-Term Dependence[†]

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Abstract

Whilst emphasis has been given to short-term dependence of financial returns, long-term dependence remains overlooked. Despite financial literature provides evidence of long-term's memory existence, serial-independence assumption prevails.

This document's long-term dependence assessment relies on rescaled range analysis *(R/S)*, a popular and robust methodology designed for Geophysics but extensively used in financial literature. Results correspond to most of the previous evidence of significant long-term dependence, particularly for small and illiquid markets, where persistence is its most common kind. Persistence conveys that the range of possible future values of the variable will be wider than the range of purely random and independent variables.

Ahead of *R/S* financial literature, authors estimate an adjusted Hurst exponent in order to properly estimate the covariance matrix at higher investment horizons, avoiding the traditional –independence reliant- square-root-of-time rule.

Ignoring long-term dependence within the mean-variance portfolio optimization results in concealed risk taking; conversely, by adjusting for long-term dependence the weight of high (low) persistence risk factors decreases (increases) as the investment horizon widens. This alleviates some well-known shortcomings of conventional portfolio optimization for long-term investors (e.g. central banks, pension funds and sovereign wealth managers), such as excessive risk taking in long-term portfolios, extreme weights, home bias, and reluctance to hold foreign currency-denominated assets.

Keywords: Portfolio optimization, Hurst exponent, long-term dependence, biased random walk, rescaled range analysis.

JEL Classification: G11, G32, G20, C14.

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Introduction

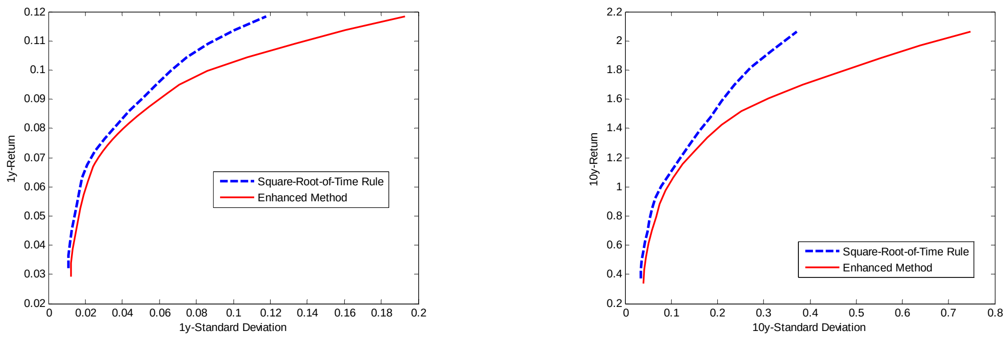
It is a widespread practice to use daily or monthly data to design portfolios with investment horizons equal or greater than a year. The computation of the annualized mean return is carried out via traditional interest rate compounding –an assumption free procedure-, whilst scaling volatility is commonly fulfilled by relying on the serial independence of returns' assumption, which results in the celebrated square-root-of-time rule.

Despite it is a well-recognized fact that the serial independence assumption for assets' returns is unrealistic at best, the convenience and robustness of the computation of the annual volatility for portfolio optimization based on the square-root-of-time rule remains largely uncontested.

As expected, the greater the departure from the serial independence assumption, the larger the error resulting from this volatility scaling procedure. Based on a global set of risk factors, the authors compare a standard mean-variance portfolio optimization (e.g. square-root-of-time rule reliant) with an enhanced mean-variance method for avoiding the serial independence assumption. Differences between the resulting efficient frontiers are remarkable, and seem to increase as the investment horizon widens (Figure 1):

Figure 1
Efficient frontiers for the standard and the enhanced methods

1-year *10-year*



Source: authors' calculations.

Because this type of error lurks beneath customary asset allocation procedures, including the prominent Black-Litterman (1992) model, the main objective of this paper is to challenge the square-root-of-time rule as a proper volatility scaling method within the mean-variance framework, and to present a robust alternative.

In order to fulfill the stated objective this paper estimates financial assets' long-run dynamic. The impact of long-term serial dependence in assets' returns is assessed for a wide set of markets and instruments, with a sample which covers the most recent market turmoil. Such estimation relies on a revised and adjusted version of the classic rescaled range analysis methodology (*R/S*) first introduced by Hurst (1951), and subsequently enhanced by Mandelbrot and Wallis (1969a and 1969b).

Similar to Hurst's results in Geophysics and to financial literature (Malevergne and Sornette, 2006; Los, 2005; Daníelsson and Zigrand, 2005), results confirm that numerous individual risk factors exhibit significant long-term dependence, thus invalidating the square-root-of-time rule. Interestingly, most previous findings related to long-term dependence in financial time-series are still supported, even after the most recent period of market crisis.

Results also demonstrate some major asset allocation issues could be explained to some extent by the inability of the square-root-of-time rule to properly scale up volatility in presence of serial long-term dependence. Some of these issues are (i) the excessive risk taking in long-term portfolios (Valdés, 2010; Reveiz *et al.* 2010; Pastor and Stambaugh, 2009; Schotman *et al.* 2008); (ii) the tendency to hold a disproportionate level of investments within the domestic market –home bias- (Solnik, 2003; Winkelmann, 2003b); (iii) the reluctance to hold foreign currency-denominated assets (Lane and Shambaugh, 2007; Davis, 2005); and (iv) the presence of extreme portfolio weights or "corner solutions" (Zimmermann *et al.* 2003; He and Litterman, 1999).

This paper consists of six chapters; this introduction is the first one. The second chapter presents a brief examination of the square-root-of-time rule and its use for scaling high-frequency volatility (e.g. daily) to low-frequency volatility (e.g. annual). The third describes and develops the classic rescaled range analysis (*R/S*) methodology for detecting and assessing the presence of long-term serial dependence of returns. The fourth chapter exhibits the results of applying classic and adjusted versions of *R/S* to selected risk factors. The fifth analyzes the consequences of the results for portfolio optimization. Finally, the last chapter highlights and discusses some relevant remarks.

1. The square-root-of-time rule

The square-root-of-time rule consists of multiplying the standard deviation calculated from a d-frequency (e.g. daily) time-series by the square-root of n, where n is the number of d units to scale standard deviation up. For example, if σ_d is the standard deviation of a d-frequency time-series, to scale volatility to an a-frequency, where $a = dn$, σ_d should be multiplied by the square-root of n, as follows:

$$\sigma_a = \sigma_{dn} = \sigma_d \sqrt[2]{n} = \sigma_d n^{0.5} \qquad \text{[F1]}$$

The value of this rule is evident for market's practitioners: as acknowledged by Dowd *et al.* (2001), obtaining time-series suitable –long enough- to make reliable volatility estimations for monthly or annual frequencies is rather difficult. Besides, even if such time-series do exist, questions about the relevance of far-in-the-past data may arise.

Perhaps the most celebrated application of the square-root-of-time rule has to do with Value at Risk (VaR) estimation. According to the technical standards originally established by the Basel Committee on Banking Supervision (BIS, 1995), the VaR must be calculated for at least a ten-day holding period. VaR estimations could be based on shorter holding periods (e.g. using daily time-series), but the ten-day holding period VaR should be attained by means of scaling up to ten days by the square-root-of-time.[1]

Discussing Bachelier's (1900) contribution to the construction of the random-walk or Brownian motion model, Mandelbrot (1963) described it as follows: if *Z(t)* is the price of a stock at the end of time period *t*, successive differences of the form *Z(t+T) - Z(t)* are (i) independent, (ii) Gaussian or normally distributed, (iii) random variables (iv) with zero mean and (v) variance proportional to the differencing interval *T*.

These assumptions have been notably challenged by mere observation of financial markets, and rejected using traditional significance tests. Nevertheless, methodologies and practices based on the Brownian motion still endure; one of such lasting practices is volatility scaling *via* the square-root-of-time rule, which is the most important prediction of the Brownian motion model (Sornette, 2003).

The assumption underneath the square-root-of-time rule is independence. Under this assumption past behavior of the variable is irrelevant. This is also known as the weak form of the Efficient Market Hypothesis (EMH), and it is the core hypothesis of the martingales model for asset pricing, which states that the current price is the best forecast for future price (Campbell *et al.*, 1997).

Under the independence assumption the probability distribution of changes in the same variable for two or more periods is the sum of the probability distribution; when two independent normal distributions are added, the result is a normal distribution in which the mean is the sum of means and the variance is the sum of variances (Hull, 2003).

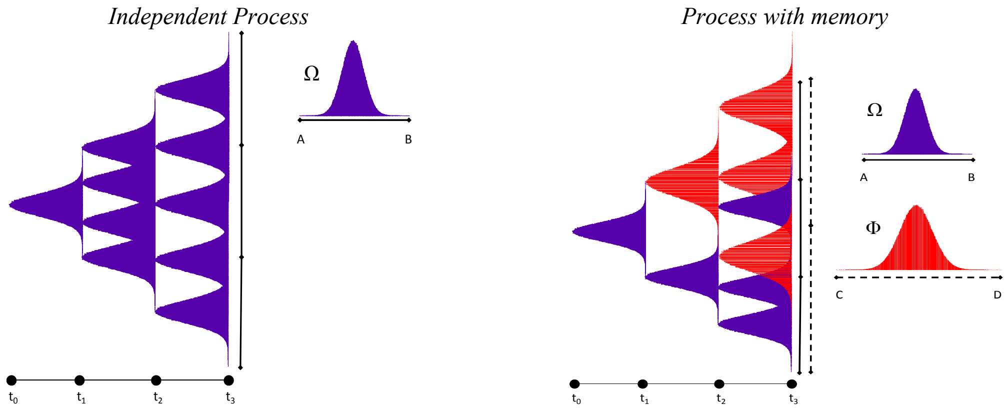
Accordingly, if the probability distribution of changes of an independent variable (Ω) has an A-B range (Figure 2, left panel), the resulting range at the end of two periods will be proportional to twice A-B, and for three periods it will be proportional to three times A-B; it is irrelevant whether the probability distribution (Ω) is Gaussian or not.

If the distribution is Gaussian the A-B range can be conveniently characterized by the variance. Hence, if the distribution can be defined as $\mathcal{N}\sim(0,1)$, where $\mathcal{N}$ stands for normally distributed, zero is the mean and 1 the variance, after three periods the distribution of the possible values of the –independent- variable corresponds to $\mathcal{N}\sim(0,1+1+1)$ or $\mathcal{N}\sim(0,3)$.

Alternatively, A-B range can be characterized by a different dispersion metric: standard deviation. However, because standard deviation corresponds to the square-root of variance, it's not additive; therefore, the three-period distribution of possible values of the –independent- variable corresponds to $\mathcal{N}\sim(0,\sqrt[2]{1+1+1})$ or $\mathcal{N}\sim(0,\sqrt[2]{3})$. This is the origin of the square-root-of-time-rule.

[1] Technical caveats to the usage of the square-root-of-time rule were recently introduced (BIS, 2009).

Figure 2
Independence and the square-root-of-time rule



Source: authors' design.

In absence of independence this rule is no longer valid. As the right panel of Figure 2 reveals, let a return above the mean lead to a different *(Φ)* more disperse distribution *(C-D>A-B)* –which is an example of dependence-, then it is impossible to affirm neither that the resulting range at the end of two periods is going to be proportional to twice *A-B*, nor twice *C-D*. This impossibility applies even if the distributions *(*Φ and Ω*)* are strictly Gaussian, and it would cause any standard rule to scale range, variance or standard deviation to falter.

Moreover, the presence of long-term dependence not only invalidates any use of the square-root-of-time rule, but contributes to explain the slow convergence of the distribution of financial assets' returns towards normality, even for low-frequency (e.g. monthly, quarterly) data (Malevergne and Sornette, 2006).

Despite asset returns' independence rests as one of the core foundations in Economics and Finance since Bachelier, contradictory evidence also dates back to the dawn of the 20th century (Mitchell, 1927; Mills, 1927; Working, 1931; Cowles and Jones, 1937).

Nevertheless, it was complex natural phenomena which forced physicists to deal with the absence of independence. It was Geophysics, not Economics nor Finance, the source of methodologies to identify and measure long-term dependence.

2. Rescaled range analysis (*R/S*)

Long-term dependence detection and assessment for time-series began with Hydrology (Mandelbrot and Wallis, 1969a), when the British physicist H.E. Hurst (1880–1978) was appointed to design a water reservoir on the Nile River. The first problem Hurst had to deal with was to determine the optimal storage capacity of the reservoir; that is, restricted to a budgetary constraint, design a dam high enough to allow for fluctuations in the water supply whilst maintaining a constant flow of water below the dam.

Deciding on the optimal storage capacity depended on the inflows of the river, which were customarily assumed to be random and independent by hydraulic engineers at that time. However, when checking Nile's historical records (622 B.C. - 1469 B.C.) Hurst discovered that flows couldn't be described as random and independent: data exhibited persistence, where years of high (low) discharges were followed by years of high dischargers (low), thus describing cycles but without an obvious periodicity.

Hurst concluded that (i) evidence contradicted the long-established independence assumption and (ii) that the absence of significant autocorrelation proved standard econometrics tests to be ineffective (Peters, 1994). Thus, since absence of independence vindicated caring about the size and sequence of flows, Hurst developed a methodology capable of capturing and assessing the type of dependence he had documented.

Hurst's methodological development was based on Einstein's (1905) work about particles' movement, which Scottish botanist Robert Brown (1828 and 1829) already depicted as inexplicable, irregular and independent. Einstein originally formulated that the distance or average displacement *(R)* covered by a particle suspended in a fluid per unit of time *(n)* followed $R = n^{0.5}$; this is analogous to the square-root-of-time rule.

Unlike Brown and Einstein, Hurst's primary objective was a broad formulae, capable of describing the distance covered by any random variable with respect to time. Hurst found his observations of several time-series were well represented by $R \sim c \times n^{H}$, where H corresponds to the way that distance *(R)* behaves with respect to time.

Hurst defined that the metric for the distance covered per unit of time or sample *(n)* would be given by the range R_n [F2], where $x_1, x_2, x_3 \ldots x_n$ correspond to the change of the random variable within the sample, and $\bar{x}_n$ is the average of these changes. Range R_n is standardized by the standard deviation of the sample for that period (S_n), which results in the rescaled range for the n sample $(R/S)_n$ [F2].

$$(R/S)_n = {R_n}/{S_n} = \frac{\left[\max\limits_{1 \le k \le n}\left(\sum_{j=1}^{k}(x_j - \bar{x}_n)\right) - \min\limits_{1 \le k \le n}\left(\sum_{j=1}^{k}(x_j - \bar{x}_n)\right)\right]}{S_n} \qquad \text{[F2]}$$

Hurst found out that the behavior of this rescaled range [F2] adequately fitted the dynamic of numerous time-series from natural phenomena, where the adjustment could be represented as follows [F3]:

$$(R/S)_n \sim c \times n^{H} \qquad \text{[F3]}$$

Paraphrasing Peters (1992), Hurst's novel methodology measures the cumulative deviation from the mean for various periods of time and examines how the range of this deviation scales over time. $\widehat{H}$, the estimated exponent that measures the way distance *(R)* behaves with respect to time, takes values within the 0 and 1 interval $(0 < \widehat{H} \leq 1)$, where $\widehat{H}=0.5$ corresponds to Einstein's and Brown's independency case.

Mandelbrot and Wallis (1969a and 1969b) proposed to plot Hurst's function [F3] for several sample sizes *(n)* in a double logarithmic scale, which served to obtain $\widehat{H}$ through a least squares regression. $\hat{H}$ would be the slope of the estimated equation [F4]; this procedure is known as the rescaled range analysis (*R/S*).

$$Log(R/S)_n = Log(c) + HLog(n) \qquad \text{[F4]}$$

According to Mandelbrot (1965) the application of *R/S* to random series with stationary and independent increases, such as those characterized by Brown (1828 and 1829) and Einstein (1905), results in $\hat{H}=0.5$, even if the distribution of the stochastic process isn't Gaussian, case in which $\hat{H}$ asymptotically converges to 0.5 $(\hat{H} \approx 0.5)$.

As said by Sun *et al.* (2007), in the $\hat{H}=0.5$ and $\hat{H} \approx 0.5$ cases the process has no memory –is independent-, hence next period's expected result has the same probability of being lower or higher than the current result. Applied to financial time-series this is akin to assuming that the process followed by assets' returns is similar to coin tossing, where the probability of heads (rise in the price) or tails (fall in the price) is the same *(½)*, and is independent of every other toss; this is precisely the theoretical base of the Capital Asset Pricing Model (CAPM), the Arbitrage Pricing Theory (APT), the Black & Scholes model and the Modern Portfolio Theory (MPT).

When $\hat{H}$ takes values between 0.5 and 1 $(0.5 < \hat{H} \leq 1)$ evidence suggests a persistent behavior, therefore one should expect the result in the next period to be similar to the current one (Sun *et al.*, 2007). According to Menkens (2007) this means that increments are positively correlated: if an increment is positive, succeeding increments are most likely to be positive than negative. In other words, each event has influence on future events; therefore there is dependence or memory in the process.

As $\hat{H}$ becomes closer to one (1) the range of possible future values of the variable will be wider than the range of purely random variables. Peters (1996) argues that the presence of persistency is a signal that today's behavior doesn't influence near future only, but distant future as well.

On the other hand, when $\hat{H}$ takes values below 0.5 $(0 \leq \hat{H} < 0.5)$ there is a signal that suggests an antipersistent behavior of the variable. This means, as said by Sun *et al.* (2007), that a positive (negative) return is more likely followed by negative (positive) ones; hence, as stated by Mandelbrot and Wallis (1969a), this behavior causes the values of the variable to tend to compensate with each other, avoiding time-series' overshooting. Applied to financial markets series, Menkens (2007) affirms that this kind of continuously compensating behavior would suggest a constant overreaction of the market, one that would drive it to a permanent adjustment process. Similarly, Peters (1996) links this behavior to the well-known "mean-reversion" process.

Hurst's methodology and results[2] were gathered, corrected and reinterpreted by Mandelbrot (1972) and Mandelbrot and Wallis (1969a and 1969b). Based on random

[2] Hurst (1956) studied 76 natural phenomena. $\hat{H}$ was significantly different from 0.5, and was close to 0.73 ($\sigma = 0.092$). Hurst found no evidence of significant autocorrelation in the first lags, which led him to reject short-term dependence as the source of this phenomenon; neither could he find a slow and gradual

simulation models they verified that (i) Hurst's conclusions were correct, but calculations were imprecise; (ii) their corrected version of *R/S* is robust to detect and measure dependence, even in presence of significant excess skewness or kurtosis[3]; (iii) their corrected version of *R/S* is asymptotically robust to short-term dependency (e.g. autoregressive and moving average processes); (iv) asymptotically $\widehat{H}=0.5$ for independent processes, even in absence of Gaussian processes; and (v) in contrast to other methodologies, *R/S* can detect non periodic cycles.

Shortcomings of Mandelbrot's (1972) and Mandelbrot and Wallis' (1969a and 1969b) developments regarding the presence of significant long-term dependence in financial time-series are depicted by Lo (1991). He introduced modified rescaled range methodology (*mR/S*) as an effort to establish whether *R/S* results are due to the presence of genuine long-term dependence, or they are due to some sort of short-term memory.

Despite considering comparative results of both *R/S* and *mR/S* as inconclusive, Los (2003) states that evidence documented by Peters (1994) shifts the balance of proof in direction of the existence of the long-term dependence in financial assets' time-series. Peters' (1994) works on long-term dependence in capital markets discarded autoregressive processes (AR), moving average (MA) and autoregressive moving average (ARMA) as sources of the persistence effect or long-term memory that is captured by the *R/S*, whilst GARCH processes showed a marginal persistence effect.[4]

Although literature about short-term dependence in assets' returns is abundant, long-term's is rather scarce, whereas *R/S* is a popular and robust methodology. As exhibited in Table 1, evidence on *R/S* application to currencies, stock indexes, fixed income securities and commodities supports the long-term dependence hypothesis, as well as Peters' (1996) statement regarding the difficulty to find antipersistent financial time-series.

Evidence of significant antipersistence has been documented for energy prices, which Weron and Przybylowicz (2000) explain as a consequence of this asset's particularities (e.g. market regulation, storage problems, transmission, distribution), and for currencies floating within a currency band that introduces non-linear features to foreign exchange trading (Reveiz, 2002).

Consequently, Peters (1996 and 1989) concluded that assets' returns don't follow a pure random walk, but exhibit some degree of persistence $(0,5 < \widehat{H} \leq 1)$; Peters named this type of tainted random walk as "biased random walk". When assets' returns follow a biased random walk they trend in one direction until some exogenous event occurs to change their bias. The presence of persistency, according to Peters, is evidence that new

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decay with increasing lags, which supported his rejection of long-term dependence in the traditional sense of Campbell *et al.* (1997).

[3] Mandelbrot and Wallis (1969a) were the first to recognize *R/S* as non-parametric, even in presence of extreme skewness or with infinite variance. León and Vivas (2010), Martin *et al.* (2003), Willinger *et al.* (1999) and Peters (1996 and 1994) verified such statement.

[4] Moreover, since the purpose of this paper is not to establish the source of dependence, either short-term or long-term, but to detect and measure its impact in financial assets' returns long-run dynamic, Lo's (1991) criticism is rather irrelevant.

events aren't immediately reflected in prices, but are manifested as an enduring bias on returns; this contradicts the EMH.

Table 1
Literature on (R/S)-estimated Hurst exponent

Author	Market	Time-Series	Period	Frequency	H	Fit
Peters (1992)	Equity	S&P500 - USA	01/1950 – 06/1988	Monthly	0,780	N/A
Ambrose *et al.* (1993)		S&P500 - USA	01/07/1962 - 31/12/1988	Daily	0,531	1,380 ‡
		S&P500 - USA	01/1950 - 07/1988	Monthly	0,622	1,490 ‡
Sierra (2007)		IPC - Mexico	04/01/1999 – 20/02/2006	Daily	0,557	0,990 §
		DJIA - USA	19/06/1999 – 22/05/2006		0,504	0,988 §
Palomas (2002)		IPC - Mexico	04/01/1988 – 11/09/2001		0,584	0,995 §
		IPC - Mexico	01/1983 – 05/2001	Monthly	0,713	0,976 §
		DJIA - USA	01/1950 - 08/2001		0,658	0,994 §
		S&P500 - USA			0,686	0,993 §
Qian and R. (2004)		DJIA - USA	11/18/1969 – 12/06/1973	Daily	0,650	N/A
Bilel and Nadhem (2009)		S&P500 - USA	03/1990 – 09/2008	Monthly	0,525	1,400 ‡
		S&PTSX - CANADA			0,541	1,465 ‡
		CAC40 - France			0,537	2,088 ‡
		DAX100 - Germany			0,541	1,644 ‡
		MIB - Italy			0,505	1,644 ‡
		NIKKEI225 - Japan			0,551	2,635 ‡
		FTSE 100 - England			0,511	2,420 ‡
Cajueiro and Tabak (2008)		NIKKEI225 - JAPAN	04/01/1999 – 30/12/2005	Daily	0,547	0,038 †
		MERVAL - Argentina			0,584	0,040 †
		BOVESPA - Brazil			0,612	0,040 †
		SENSEX - India			0,591	0,040 †
		KOSPY - Korea			0,551	0,039 †
		IPSA - Chile			0,594	0,040 †
		IPC - Mexico			0,557	0,039 †
		IGBVL - Peru			0,656	0,042 †
		ISE - Turkey			0,538	0,036 †
		TA-100 - Israel			0,584	0,041 †
		FTSE100 - England			0,521	0,039 †
		S&P500 - USA			0,519	0,037 †
Jagric *et al.* (2005)		PX50 - Czech Republic	07/09/1993 – 03/07/2004		0,645	0,018 †
		BUX - Hungary	02/01/1991 – 30/06/2004		0,626	0,015 †
		WSE - Poland	18/03/1994 – 03/08/2004		0,569	0,018 †
		RTS - Russia	01/09/1995 – 05/08/2004		0,648	0,020 †
		SAX - Slovakia	03/07/1995 – 30/07/2004		0,525	0,020 †
		SBI - Slovenia	07/01/1993 – 16/07/2004		0,656	0,017 †
McKenzie (2001)		Australian Stock exchange	04/1876 – 03/1996		0,571	2,027 ¤
				Monthly	0,622	1,850 ¤
Alptekin (2008)	Commodities	Gold - Istambul Gold Exchange	01/01/2003 – 17/03/2008	Daily	0,600	2,100 ‡
Corazza *et al.* (1997)		Corn Futures - CBOT	02/01/1981 – 24/10/1991		0,760	N/A
		Oats Futures - CBOT			0,700	N/A
		Soybean Futures - CBOT			0,740	N/A
		Soybean oil futures - CBOT			0,800	N/A
		Wheat futures - CBOT			0,650	N/A
Erzgraber *et al.* (2008)		Energy (NordPool) - Norway	01/01/1999 - 26/01/2007		0,270	N/A
Weron and Przandbandlowicz (2000)		Energy (CalPX) California	01/03/1998 - 31/01/2000	Hourly	0,439	N/A
		Energy (SWEP) - Switzerland	11/03/1998 -31/03/2000	Daily	0,529	N/A
Batten *et al.* (1999)	Currencies	DMK/USD	17/01/1976 - 03/09/1998		0,623	2,248 ¤
		CHF/USD			0,610	2,053 ¤
		JPY/USD			0,609	1,954 ¤
		GBP/USD			0,590	1,487 ¤
Sierra (2007)		MXN/USD	02/01/1995 – 14/02/2006		0,526	0,994 §
		USD/EUR	19/06/1999 – 22/05/2006		0,559	0,995 §
Da Silva *et al.* (2007)		BRL/USD	02/01/1995 – 31/08/2006		0,630	3,260 ¤
Souza *et al.* (2008)		DEM/USD	27/05/1986 – 31/12/1998		0,580	0,026 †
		3 months future DEM/USD			0,571	0,026 †
		FRF/USD			0,576	0,026 †
		GBP/USD			0,567	0,026 †
		ITL/USD			0,598	0,026 †
Peters (1992)	Fix.Income	30 years Treasuries - USA	01/1950 – 06/1988	Monthly	0,670	N/A

§ Corresponds to the R^2 of the regression [F4]
† Corresponds to the standard error of estimated H
‡ Corresponds to Lo's Vq statistic (1991)
¤ Corresponds to the t statistic by Couillard and Davison (2005)

Source: authors' design.

Some explanations for financial assets' return persistence are found in human behavior, since the latter contradicts rationality assumption in several ways, for example: (i) investors' choices are not independent, and they are characterized by non-linear and imitative behavior (LeBaron and Yamamoto, 2007; Sornette, 2003); (ii) investors resist to change their perception until a new credible trend is established (Singh and Dey, 2002; Peters, 1996), and (iii) investors don't react to new information in a continuous manner, but rather in a discrete and cumulative way (Singh and Dey, 2002).

Other explanations for financial assets' return persistence have to do with the importance of economic fundamentals (Nawrocki, 1995; Lo, 1991; Peters, 1989), and the use of privileged information (Menkens, 2007). Alternatively, some authors (Bouchaud *et al.*, 2008; Lillo and Farmer, 2004), based on the persistence of the number and volume of buying and selling orders in transactional systems, conclude that markets' liquidity make instantaneous trading impossible, leading to transactions' splitting, and decisions' clustering, resulting in market prices which don't fully reflect information immediately, but incrementally.

3. Estimated Hurst exponent ($\widehat{H}$) for major risk factors

Estimating the Hurst exponent ($\widehat{H}$) requires the implementation of the algorithm described in the Appendix, and the design of significance tests for evaluating the null hypothesis of independence.

a. Confidence intervals and significance tests

One of the main difficulties of *R/S* methodology is the selection of an *ad-hoc* optimal size of periods (n) to calculate $(R/S)_n$. In the literature there is consensus about *R/S* being not reliable for reduced periods because estimations may become unstable and biased (Cannon *et al.,* 1997; Peters, 1994; Ambrose *et al.*, 1993). However, there is no consensus about an optimal minimum size of periods (n_{min}).[5]

The same issue arises with the choice of optimal maximum period size (n_{max}). Cannon *et al.* (1997) and Peters (1996) recognize that the stability of $\widehat{H}$ diminishes when using extended periods. Therefore, Cannon *et al.* advise to dismiss the use of data windows where estimations are made on a few segments of the time-series.

Given the absence of consensus on the optimal period size, all calculations were made using a minimum size of 32 observations ($n_{min} \geq 2^5$). This choice not only recognizes the intricacy of finding extended time-series (Peters, 1994), but also results in reduced standard errors of the estimators in the sense of Cannon *et al.* (1997), and guarantees that the effect of conventional short-term serial dependence (e.g. autocorrelation) for a daily-frequency series is minimized (Nawrocki, 1995).

The maximum period size constrain (n_{max}) consists of restricting time-series to be divided at least in ten contiguous non-overlapping segments; in this way estimations based on a narrow number of samples and unstable estimators are avoided.

[5] Cannon *et al.* (1997) estimate optimal minimum size of periods to be $n_{min} \geq 2^8$ ($\geq$ 256 observations) to achieve standard deviations below 0.05; Mandelbrot and Wallis (1969a) use 20 observations; Wallis and Matalas (1970) point out that the window must have at least 50 observations, unless series are of considerable length; Peters (1994) acknowledges that financial series are not long enough to discard reduced windows, and suggests at least 10 observations; Nawrocki (1995) argues that minimum number of observations should be large enough to minimize the effect of short-term dependence.

Concerning significance tests for $\widehat{H}$, two well-documented issues have to be taken into account (León and Vivas, 2010; Ellis, 2007; Couillard and Davison, 2005; Peters, 1994). First, there is a positive bias in the estimation –overestimation- of H resulting from finite time-series and minimum size of periods below approximately 1.000 observations. Second, $\widehat{H}$ for normal and non-normal distributed random variables distribute like a normal.

Regarding the first issue, the estimation bias resulting in the overestimation of $\widehat{H}$ can be conveniently assessed. Several assessment methods for estimating such bias have been documented, but this work focuses on the single most well-known. First proposed by Anis and Lloyd (1976), subsequently revised by Peters (1994), and recently verified and applied by León and Vivas (2010), Ellis (2007) and Couillard and Davison (2005), the chosen method consists of a functional approximation for estimating the expected value of $(R/S)_n$ when the random variable is independent and of finite length. This method yields the expected Hurst exponent corresponding to an independent random variable, which will be noted as $\dot{H}$, and is based on the following calculation of the expected value of $(R/S)_n$:

$$E(R/S)_n = \frac{n-\frac{1}{2}}{n}\frac{1}{\sqrt{n\pi/2}}\sum_{i=1}^{n-1}\sqrt{\frac{n-i}{i}} \qquad \text{[F5]}$$

Any divergence of $\widehat{H}$ from $\dot{H}$ would signal the presence of long-term memory in time-series. However, as customary in statistical inference, it is critical to develop appropriate statistical tests to distinguish between significant and non-significant deviations from long-term independence null hypothesis.

The significance test used is similar to those proposed by Ellis (2007) and Couillard and Davison (2005). Because $\widehat{H}$'s distribution is established to be normal, even for random variables that are not, a conventional *t*-statistic test may be implemented. Let $\widehat{H}$ be the *R/S*'s estimated value of the Hurst exponent, $\hat{\mu}(\dot{H})$ and $\hat{\sigma}(\dot{H})$ the expected value and standard deviation of the expected Hurst exponent corresponding to an independent random variable ($\dot{H}$), the significance test would be as follows:[6]

$$t = \frac{\widehat{H} - \hat{\mu}(\dot{H})}{\hat{\sigma}(\dot{H})} \qquad \text{[F6]}$$

As usual, if t is higher than ±1.96 it is possible to reject the null hypothesis of long-term independence with a 95% confidence level. The sign of t reveals the type of dependence: if it is positive (negative) there is evidence of persistence (antipersistence).

For convenience, given that $\dot{H}$ is the estimated Hurst exponent for random, independent and finite time-series of length N, the spread between $\dot{H}$ and 0.5 corresponds to the bias

[6] Let N be the length of time-series, due to $\widehat{H}$ distributing like a normal the ordinary choice for $\hat{\sigma}(\dot{H})$ is $\approx 1/N^{1/2}$ as in Peters (1994). According to Couillard and Davison (2005) this choice corresponds to infinite length time-series, and yields easy and frequent rejections of the independence null hypothesis. They propose $\approx 1/eN^{1/3}$, which is authors' choice.

estimation resulting from using finite time-series and the choice of the size of periods *(n)*. Subtracting such spread from the Hurst exponent estimated using *R/S*, namely $\widehat{H}$, results in an adjusted estimated Hurst exponent, which will be noted as $\breve{H}$:

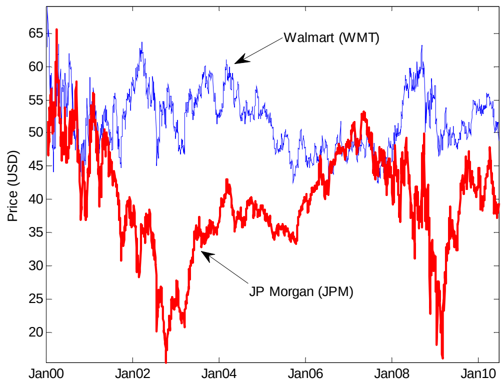
$$\breve{H} = \widehat{H} - \left(\dot{H} - 0.5\right) \qquad [F7]$$

This adjusted estimated Hurst exponent ($\breve{H}$) is essential since it allows a practical and unbiased volatility scaling as will be presented in the following sections.

b. Estimated values of Hurst exponent ($\widehat{H}$)

Figure 3 exhibits the Walmart and JP Morgan's price-series from January 1st 2000 to June 25th 2010.

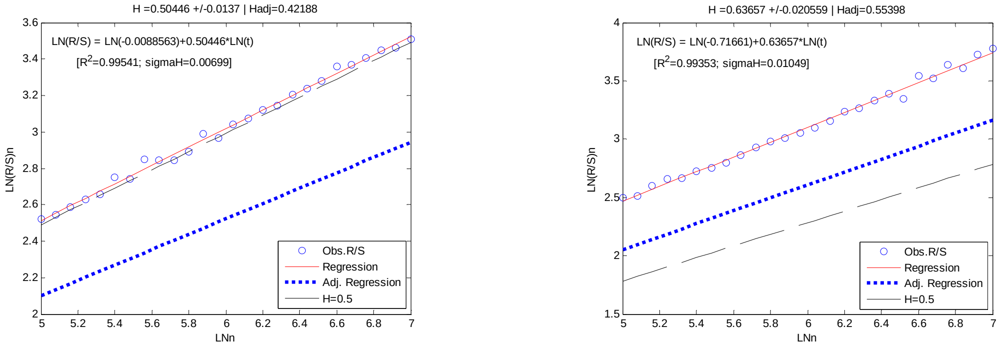
Figure 3
Daily prices for Walmart and JP Morgan



Source: authors' calculations.

Walmart's exhibits a narrower range in which prices fluctuate, where returns appear to compensate each other, whilst JP Morgan's appear to persist overtime; since both share the same dollar-scale, it is somewhat intuitive that JP Morgan's time-series are more persistent than Walmart's. Figure 4 exhibits the graphical result of applying *R/S*.

Figure 4
Walmart and JP Morgan (adjusted and unadjusted Hurst exponent)
Walmart (WMT) *JP Morgan (JPM)*



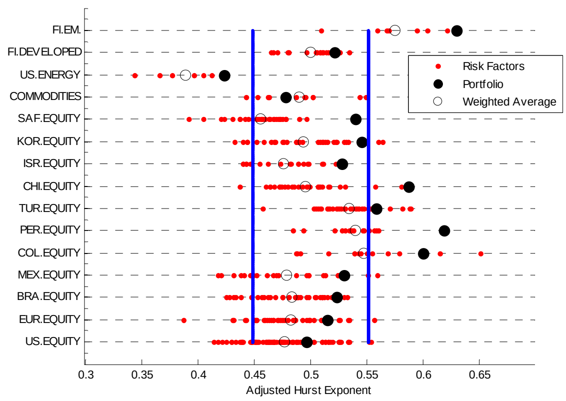
Source: authors' calculations.

Walmart exhibits an estimated Hurst exponent slightly above 0.5 ($\widehat{H}_{WMT} = 0.504$), which would be a signal of non-significant persistence, whilst JP Morgan's $\widehat{H}$ clearly diverges from 0.5 ($\widehat{H}_{JPM} = 0.637$). Nevertheless, after acknowledging the positive estimation bias, adjusted estimated Hurst exponent reveals that Walmart's time-series is in fact antipersistent ($\breve{H}_{WMT} = 0.422$), whereas JP Morgan's remains as being persistent ($\breve{H}_{JPM} = 0.554$).

Figure 5 exhibits the adjusted estimated Hurst exponent ($\breve{H}$) for individual risk factors (small dots) pertaining to distinct markets (e.g. developed and emerging) and diverse instruments (fixed income, equity and commodities). As before, if the adjusted estimated Hurst exponent ($\breve{H}$) is greater (lower) than 0.5 there exists evidence of persistence (antipersistence), where the area between the vertical lines correspond to the 95% confidence interval in which the independence hull hypothesis can't be rejected.

Individual risk factors across markets and instruments display different degrees of dependence, where persistence is typical of emerging markets' fixed income instruments (FI.EM) and of less-developed equity markets (e.g. Colombia, Turkey and Peru). Developed equity markets (e.g. US and EUR) and liquid emerging markets (e.g. Brazil, Mexico) show less incidence of persistent individual risk factors, even with several cases of antipersistence. These findings support Cajueiro and Tabak's (2008) comparison between developed and emerging markets.

Figure 5
Adjusted Estimated Hurst Exponent ($\breve{H}$)[7]



Source: authors' calculations.

The results also correspond to the findings of Weron and Przybylowicz (2000) in relation with significant antipersistence of energy prices, but contradict Peters' (1996) affirmation about the difficulty to find financial time-series with antipersistent returns.

[7] The markets included are the following: FI.EM. – Emerging Markets' Fixed Income (EMBI Global of Brazil, Mexico, Colombia, Peru, South Africa, Turkey and Chile); FI.DEVELOPED – Developed Markets' Fixed Income (as in Table 2); US.ENERGY (Off-peak day ahead electricity for several US regions); COMMODITIES (oil, gold, copper, wheat, corn, cotton, aluminum, sugar, coffee, cocoa, rice, soy); and a market-capitalization representative set of securities from the equity markets of the United States (US), Europe (EUR), Brazil (BRA), Mexico (MEX), Colombia (COL), Peru (PER), Turkey (TUR), Chile (CHI), Israel (ISR), Korea (KOR) and South Africa (SAF). All estimations were based on January 1st 2000-June 25th 2010 time-series, except US.ENERGY (January 1st 2002- June 25th 2010).

Regarding persistence at the portfolio level, Figure 5 displays the adjusted estimated Hurst exponent ($\breve{H}$) for an equally weighted portfolio of the individual risk factors (filled circles) and the equally weighted average of the individual risk factor's adjusted estimated Hurst exponent (empty circles). It is remarkable that the portfolios' adjusted estimated Hurst exponent tends to be higher than the weighted average of the individual exponents, which would indicate that diversification effect does not apply to serial dependence as it does to variance or standard deviation.

It is also noteworthy that for emerging fixed income and equity markets the portfolios' adjusted estimated Hurst exponent ($\breve{H}$) rests significantly higher than the weighted average of the individual exponents. Because aggregating risk factors should result in specific or idiosyncratic risk diversification, this could indicate that the remaining systemic risk is relatively more important for emerging than for developed markets; this could be the result of poor diversification opportunities within a small and illiquid market, or of the generalized impact of systemic shocks and the corresponding changes in risk appetite and liquidity in those markets.

4. Portfolio optimization under long-term dependence

The most far-reaching consequences of long-term dependence or memory in financial assets' returns were pointed out by Lo (1991). He recognized that the long-term dependence conveys the invalidity of modern Finance's milestones, where the most hard-hit would be the optimal consumption/savings and portfolio decisions, as well as the pricing of derivatives based on martingale methods.

a. Volatility scaling, investment decisions and portfolio optimization

Conventional portfolio optimization uses high-frequency data and customary procedures for return and volatility scaling in order to obtain allocations for low-frequency horizons.

Let $\hat{\mu}_d$ and $\hat{\sigma}_d$ be the estimated high-frequency (e.g daily) continuously compounded expected return and standard deviation, $\hat{\mu}_a$ and $\hat{\sigma}_a$ the estimated low-frequency (e.g. annual) continuously compounded expected return and standard deviation, and p the number of days-in-a-year convention. The standard procedure for asset allocation typically involves the following expected return [F8] and volatility escalation [F9]:

$$\hat{\mu}_a = \sum_{t=1}^{p} \hat{\mu}_{d(t)} \quad \text{[F8]}$$

$$\hat{\sigma}_a = \hat{\sigma}_d p^{0.5} \quad \text{[F9]}$$

The standard procedure to scale returns up (e.g. from daily to annual) is assumption-free, and consists of interest compounding calculations. However, conventional volatility scaling inexorably involves the serial independence assumption.

If assets' returns exhibit no serial dependence using the square-root-of-time rule is adequate. Nevertheless, in absence of independence some assets' volatility may increase with time horizon, while others' may decrease; even if all assets' volatility increases, it may not increase at the same pace. Thus, Holton (1992) highlights the importance of considering volatility and investment horizon as risk's first and second dimensions.

In presence of long-term dependence scaling returns up as in [F8] remains unchanged. But for estimating volatility the scaling procedure should be generalized as follows:

$$\hat{\sigma}_a = \hat{\sigma}_d p^{\breve{H}} \qquad \text{[F10]}$$

Additionally, because mean-variance portfolio optimization involves working with the covariance matrix, the latter should be scaled up properly. Under the random-walk assumption low-frequency covariance between two assets, i and j, corresponds to the arithmetic sum of high-frequency covariances (Winkelmann, 2003); thus the relative variance between assets remain unrelated to the investment horizon.

Nevertheless, in presence of dependence, either $\breve{H}_i \neq 0.5$ or $\breve{H}_j \neq 0.5$, as an extension to the volatility scaling procedure [F10], the *d*-frequency covariance between assets i and j ($\hat{\sigma}^2_{\{(i,j),d\}}$) should be scaled up to the *a*-frequency covariance ($\hat{\sigma}^2_{\{(i,j),a\}}$) as in Greene and Fielitz (1979) [F11]; this recognizes that memory in financial time-series cause relative variance between assets to vary with the investment horizon.

$$\hat{\sigma}^2_{\{(i,j),a\}} = \left(p^{\breve{H}_i+\breve{H}_j}\right)\left(\hat{\sigma}^2_{\{(i,j),d\}}\right) \qquad \text{[F11]}$$

b. Long-term dependence inclusive portfolio optimization

In order to illustrate the impact of including long-term dependence adjustments to the covariance matrix scaling for asset allocation, a long-term portfolio optimization exercise is implemented based on the two methods for scaling volatility: (i) the square-root-of-time rule [F9] conventional method, and (ii) the method proposed by the authors [F10 and F11].

The square-root-of-time rule based method begins by estimating the first two moments of the distribution of the risk factors and the covariance matrix from daily data. Afterwards a traditional mean-variance optimization is employed, and the expected return and standard deviation of the resulting portfolios are customarily scaled up; since the square-root-of-time rule assumes volatilities' time-consistency the portfolio weights remain the same regardless of the investment horizon.

The second method also begins by estimating the first two moments of the distribution and the covariance matrix from daily data. Next, because risk factors' dependence causes portfolio weights to vary according to the investment horizon, the standard deviation and covariance matrix scaling for long-term dependence effects [F10 and F11] takes place before optimizing.

Table 2 presents the set of risk factors to be considered in the portfolio optimization procedure. Consistent with literature on strategic asset allocation, which points out that currency risk hedging is inappropriate for long-term portfolios (Solnik *et al.*, 2003; Froot, 1993), all risk factors were included in their original currency.

Table 2
*Adjusted Hurst exponents for risk factors**

MARKET	DESCRIPTION	CURRENCY	SOURCE	ABBREVIATION	EXPECTED RETURN	STD. DEVIATION	RETURN/ RISK	Adjusted Ȟ	*t-stat*
COMMODITIES	PRECIOUS METALS	USD	S&P	PREC.MET.	0,03%	1,07%	0,030	0,47	(1,15)
	INDUSTRIAL METALS			IND.MET.	0,02%	1,38%	0,013	0,48	(1,00)
	AGRICULTURE & LIVE STOCK			AGR.&L.S.	-0,01%	0,91%	(0,015)	0,50	(0,11)
	CRUDE OIL			CRUDE.OIL	0,04%	2,24%	0,017	0,48	(0,92)
EQUITY	DEVELOPED MARKETS		MSCI	EQ.DEV.	0,01%	1,01%	0,014	0,51	0,51
	EMERGING MARKETS			EQ.EM	0,02%	1,26%	0,013	0,59	**3,70**
FIXED INCOME	EMERGING MARKETS		JPM	EMBI	0,04%	0,74%	0,060	0,59	**3,86**
	US.TREASURY 1-5Y		MERRILL LYNCH / BofA	US.T 1-5Y	0,02%	0,15%	0,142	0,53	1,14
	US.TREASURY 5-10Y			US.T 5-10Y	0,03%	0,36%	0,075	0,52	0,77
	US.TREASURY 10+Y			US.T 10+Y	0,03%	0,60%	0,052	0,51	0,26
	US.CORP AAA-AA 1-5Y			US.CORP 1-5Y	0,02%	0,17%	0,136	0,52	0,72
	US.CORP AAA-AA 5-10Y			US.CORP 5-10Y	0,03%	0,37%	0,073	0,50	0,05
	US.CORP AAA-AA 10+Y			US.CORP 10+Y	0,03%	0,55%	0,053	0,48	(0,94)
	US.MORTGAGES AAA			US.MRTG	0,03%	0,21%	0,126	0,50	(0,20)
	GER.TREASURY 1-5Y	EUR		GER.T 1-5Y	0,02%	0,12%	0,164	0,54	1,69
	GER.TREASURY 5-10Y			GER.T 5-10Y	0,03%	0,27%	0,098	0,52	0,84
	GER.TREASURY 10+Y			GER.T 10+Y	0,03%	0,53%	0,063	0,48	(0,87)
	JAP.TREASURY 1-5Y	JPY		JAP.T 1-5Y	0,01%	0,09%	0,083	0,51	0,42
	JAP.TREASURY 5-10Y			JAP.T 5-10Y	0,02%	0,24%	0,062	0,50	(0,21)
	JAP.TREASURY 10+Y			JAP.T 10+Y	0,02%	0,41%	0,047	0,50	(0,05)
	UK.TREASURY 1-5Y	GBP		UK.T 1-5Y	0,02%	0,14%	0,179	0,55	**2,10**
	UK.TREASURY 5-10Y			UK.T 5-10Y	0,03%	0,30%	0,096	0,54	1,55
	UK.TREASURY 10+Y			UK.T 10+Y	0,03%	0,52%	0,057	0,49	(0,28)

(*) Calculations based on daily time-series (Jan 1st 1995 – June 25th 2010). Significant (95%) *t-stat* results are highlighted.
Source: authors' calculations.

According to Table 2 long-term dependence is significant for the two emerging market's risk factors considered, namely equity and fixed income indexes, which – again- validates the findings of Cajueiro and Tabak (2008).

Regarding commodities, divergence between $\breve{H}$ and 0.5 is rather low, with minor signals of antipersistence for metals and crude oil; agriculture and live stock commodities' $\breve{H}$ matches the independence assumption.

Developed markets' fixed income risk factors show low levels of persistence, except for short-term treasuries from UK and Germany, and medium-term treasuries from UK; it is noteworthy that long-term fixed instruments consistently tend to exhibit lower persistence than short-term ones. Concerning developed markets' equity, findings of Cajueiro and Tabak (2008), Menkens (2007), Couillard and Davison (2004), Ambrose *et al.* (1993) and Lo (1991) are verified: there is no evidence of significant long-term dependence; therefore, Peters (1992) findings about long-term dependence in developed markets are contradicted.

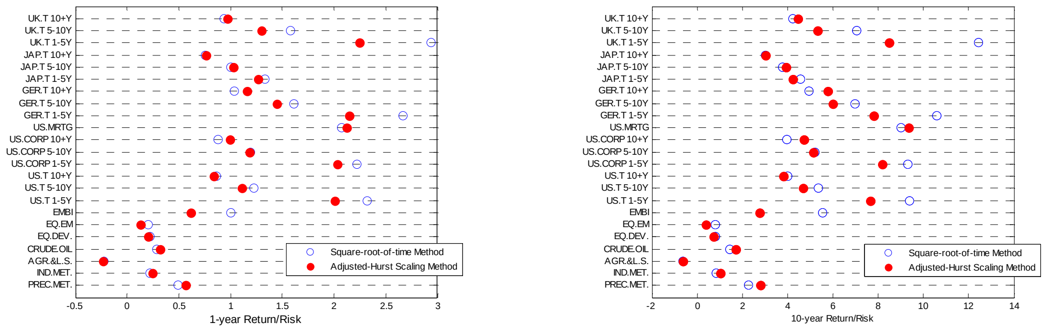
Interestingly, contrary to conventional wisdom, fixed income instruments' mean returns significantly outperformed equity's for the time-series under analysis; thus, it is likely that resulting efficient portfolios will disregard equity *vis-à-vis* academic basics. This supports recent concerns regarding the existence of a *natural hedge* from stocks in the long-run and of a positive equity risk premium (Valdés, 2010; Arnott, 2009).

Using the adjusted estimated Hurst exponent (Table 2) Figure 6 exhibits the risk/return ratios for both scaling methods for 1-year and 10-year investment horizons. Relative return/risk ratios between methods clearly differ for almost all risk factors. Once dependence is taken into account extreme differences between return/risk ratios due to concealed riskiness resulting from serial-dependence are moderated; hence, it is plausible that adjusting for long-term persistence helps mitigating the well-known tendency of mean-variance optimization to provide extreme weights or corner solutions. Figure 6 results concur with Greene and Fielitz's (1979) concern about how return/risk performance measures (e.g. Sharpe, Treynor and Jensen ratios) are affected by the differencing interval assumption in presence of long-term dependence.

Figure 6

Return/Risk ratio for the standard and the enhanced methods

1-year *10-year*



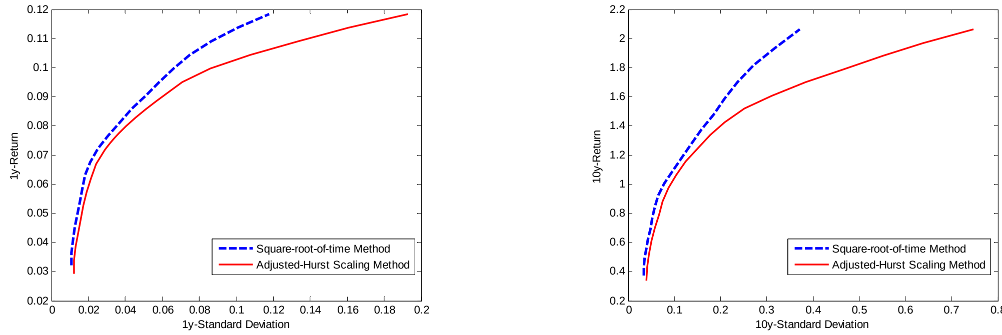
Source: authors' calculations.

Figure 7 exhibits the efficient frontiers for both scaling methods for 1-year and 10-year investment horizons. As expected, the standard method obtains a strictly dominating frontier with higher levels of return for each level of risk.

Figure 7

Efficient frontiers for the standard and the enhanced methods

1-year *10-year*



Source: authors' calculations.

Strict dominance of the traditional method's efficient frontier occurs because relative return/risk ratios do not change with time horizon; adjusting for long-term dependence causes efficient portfolio weights associated with high (low) persistence risk factors to decrease (increase) as the horizon increases. This statement becomes evident when observing portfolio weights obtained by each method along the 1-year horizon frontier (Table 3). Each frontier consists of twenty portfolios, from the lowest risk to the highest return; average adjusted exponent ($\breve{H}$) and average return/risk ratio for each category of risk factors are also reported.

Table 3
1-year horizon efficient frontier's weights

(Panel a.) Square-root-of-time method

Port. #	RETURN / RISK	COMMODITES	EMERGING MARKETS		DEVELOPED MARKETS							
			EQUITY	EMBI	EQUITY	US.TREAS	US.CORP	US.MRTG	GER.TRES	JAP.TREAS	UK.TREAS	
		0,482	**0,585**	**0,589**	**0,512**	**0,517**	**0,499**	**0,495**	**0,513**	**0,501**	**0,526**	*←Adj. H exponent*
		0,192	**0,207**	**1,006**	**0,221**	**1,469**	**1,433**	**2,073**	**1,773**	**1,031**	**1,817**	*←Return/Risk*
1	3,0	0,9%	0,9%	0,0%	0,9%	11,0%	0,0%	0,0%	18,7%	58,4%	9,1%	
2	3,3	0,7%	1,0%	0,0%	1,1%	13,1%	0,0%	0,0%	19,2%	48,9%	16,1%	
3	3,5	0,7%	1,0%	0,3%	1,2%	15,0%	0,0%	0,0%	19,5%	39,3%	22,9%	
4	3,6	0,9%	0,9%	0,8%	1,3%	14,5%	2,2%	0,0%	19,9%	30,1%	29,5%	
5	3,6	1,0%	0,8%	1,3%	1,3%	11,8%	4,9%	1,4%	20,2%	21,4%	35,9%	
6	3,6	1,1%	0,8%	1,5%	1,4%	11,2%	6,5%	2,1%	21,6%	13,4%	40,3%	
7	3,5	1,2%	0,8%	2,0%	1,4%	6,9%	9,5%	4,4%	20,5%	6,8%	46,5%	
8	3,4	1,6%	0,3%	3,3%	1,1%	0,0%	12,1%	10,0%	10,2%	3,3%	58,1%	
9	3,3	2,1%	0,0%	6,2%	0,0%	0,0%	3,1%	19,3%	0,0%	1,8%	67,4%	
10	2,9	2,6%	0,0%	11,8%	0,0%	0,0%	0,0%	24,7%	1,8%	0,0%	59,1%	
11	2,5	3,5%	0,0%	16,8%	0,0%	0,0%	0,0%	26,3%	8,0%	0,0%	45,4%	
12	2,2	4,3%	0,0%	20,9%	0,0%	0,0%	0,0%	29,4%	9,8%	0,0%	35,7%	
13	2,0	5,0%	0,0%	25,1%	0,0%	0,0%	0,0%	31,9%	11,7%	0,0%	26,3%	
14	1,8	5,7%	0,0%	31,6%	0,0%	0,0%	0,0%	20,5%	18,7%	0,0%	23,4%	
15	1,6	6,5%	0,0%	37,8%	0,0%	2,5%	0,0%	7,4%	24,4%	0,0%	21,4%	
16	1,5	7,1%	0,0%	44,4%	0,0%	4,9%	0,0%	0,0%	31,7%	0,0%	11,9%	
17	1,4	6,4%	0,0%	53,5%	0,0%	2,1%	0,0%	0,0%	38,0%	0,0%	0,0%	
18	1,3	3,0%	0,0%	67,9%	0,0%	0,0%	0,0%	0,0%	29,1%	0,0%	0,0%	
19	1,1	2,5%	0,0%	83,5%	0,0%	0,0%	0,0%	0,0%	14,0%	0,0%	0,0%	
20	1,0	0,0%	0,0%	100,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	

(Panel b). Adjusted-Hurst scaling method

Port. #	RETURN / RISK	COMMODITES	EMERGING MARKETS		DEVELOPED MARKETS							
			EQUITY	EMBI	EQUITY	US.TREAS	US.CORP	US.MRTG	GER.TRES	JAP.TREAS	UK.TREAS	
		0,482	**0,585**	**0,589**	**0,512**	**0,517**	**0,499**	**0,495**	**0,513**	**0,501**	**0,526**	*←Adj. H exponent*
		0,227	**0,129**	**0,615**	**0,207**	**1,317**	**1,403**	**2,127**	**1,586**	**1,019**	**1,503**	*←Return/Risk*
1	2,4	1,2%	0,5%	0,0%	1,1%	11,9%	0,0%	0,0%	14,8%	66,0%	4,6%	
2	2,8	1,0%	0,5%	0,0%	1,2%	11,5%	0,6%	2,8%	15,7%	56,8%	9,9%	
3	2,9	1,2%	0,5%	0,0%	1,3%	5,8%	4,4%	7,5%	16,4%	48,2%	14,9%	
4	3,0	1,5%	0,4%	0,0%	1,3%	0,0%	8,1%	12,4%	17,0%	39,2%	19,9%	
5	3,0	1,9%	0,4%	0,0%	1,4%	0,0%	8,6%	16,0%	17,3%	29,5%	24,7%	
6	3,0	2,1%	0,5%	0,0%	1,6%	0,0%	9,6%	18,2%	18,7%	21,4%	27,9%	
7	3,0	2,4%	0,6%	0,0%	1,7%	0,0%	10,6%	20,4%	20,1%	13,2%	31,1%	
8	2,9	3,1%	0,3%	0,7%	1,4%	0,0%	8,1%	28,8%	9,6%	7,4%	40,6%	
9	2,8	4,1%	0,0%	2,1%	0,4%	0,0%	0,0%	41,7%	0,0%	3,7%	47,9%	
10	2,5	5,3%	0,0%	4,2%	0,0%	0,0%	0,0%	46,4%	8,0%	0,0%	36,1%	
11	2,2	7,1%	0,0%	6,6%	0,0%	0,0%	0,0%	49,6%	19,1%	0,0%	17,6%	
12	1,9	8,9%	0,0%	9,0%	0,0%	0,0%	0,0%	51,8%	30,4%	0,0%	0,0%	
13	1,7	10,7%	0,0%	12,9%	0,0%	0,0%	0,0%	34,1%	42,3%	0,0%	0,0%	
14	1,5	12,6%	0,0%	16,8%	0,0%	0,7%	0,0%	16,0%	53,9%	0,0%	0,0%	
15	1,3	13,5%	0,0%	21,9%	0,0%	2,4%	0,0%	0,0%	62,2%	0,0%	0,0%	
16	1,2	8,2%	0,0%	35,1%	0,0%	0,0%	0,0%	0,0%	56,7%	0,0%	0,0%	
17	1,0	8,1%	0,0%	49,9%	0,0%	0,0%	0,0%	0,0%	41,9%	0,0%	0,0%	
18	0,8	9,0%	0,0%	65,0%	0,0%	0,0%	0,0%	0,0%	26,0%	0,0%	0,0%	
19	0,7	9,9%	0,0%	80,2%	0,0%	0,0%	0,0%	0,0%	10,0%	0,0%	0,0%	
20	0,6	0,0%	0,0%	100,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	

Source: authors' calculations.

Relative overweighting of persistent risk factors (e.g. emerging markets' fixed income - EMBI) is evident for the conventional method (Panel a.). When dependence is taken into account such overweight diminishes in favor of near-independent or antipersistent risk factors, such as Japan and German treasuries, US Mortgages or commodities. Such persistent risk factors' relative overweighting is also validated for the ten-year horizon (Table 4).

Table 4
10-year horizon efficient frontier's weights

(Panel a.) Square-root-of-time method

Port. #	RETURN / RISK	COMMODITIES	EMERGING MARKETS EQUITY	EMERGING MARKETS EMBI	DEVELOPED MARKETS EQUITY	US.TREAS	US.CORP	US.MRTG	GER.TRES	JAP.TREAS	UK.TREAS	
		0,482	**0,585**	**0,589**	**0,512**	**0,517**	**0,499**	**0,495**	**0,513**	**0,501**	**0,526**	*←Adj. H exponent*
		0,993	**0,793**	**5,542**	**0,821**	**6,251**	**6,162**	**9,000**	**7,518**	**3,786**	**7,906**	*←Return/Risk*
1	10,9	0,9%	0,9%	0,0%	0,9%	11,0%	0,0%	0,0%	18,7%	58,4%	9,1%	
2	12,4	0,7%	1,0%	0,0%	1,1%	13,1%	0,0%	0,0%	19,2%	48,9%	16,1%	
3	13,4	0,7%	1,0%	0,3%	1,2%	15,0%	0,0%	0,0%	19,5%	39,3%	22,9%	
4	14,0	0,9%	0,9%	0,8%	1,3%	14,5%	2,2%	0,0%	19,9%	30,1%	29,5%	
5	14,3	1,0%	0,8%	1,3%	1,3%	11,8%	4,9%	1,4%	20,2%	21,4%	35,9%	
6	14,4	1,1%	0,8%	1,5%	1,4%	11,2%	6,5%	2,1%	21,6%	13,4%	40,3%	
7	14,6	1,2%	0,8%	2,0%	1,4%	6,9%	9,5%	4,4%	20,5%	6,8%	46,5%	
8	14,5	1,6%	0,3%	3,3%	1,1%	0,0%	12,1%	10,0%	10,2%	3,3%	58,1%	
9	14,1	2,1%	0,0%	6,2%	0,0%	0,0%	3,1%	19,3%	0,0%	1,8%	67,4%	
10	12,8	2,6%	0,0%	11,8%	0,0%	0,0%	0,0%	24,7%	1,8%	0,0%	59,1%	
11	11,3	3,5%	0,0%	16,8%	0,0%	0,0%	0,0%	26,3%	8,0%	0,0%	45,4%	
12	10,1	4,3%	0,0%	20,9%	0,0%	0,0%	0,0%	29,4%	9,8%	0,0%	35,7%	
13	9,3	5,0%	0,0%	25,1%	0,0%	0,0%	0,0%	31,9%	11,7%	0,0%	26,3%	
14	8,6	5,7%	0,0%	31,6%	0,0%	0,0%	0,0%	20,5%	18,7%	0,0%	23,4%	
15	8,0	6,5%	0,0%	37,8%	0,0%	2,5%	0,0%	7,4%	24,4%	0,0%	21,4%	
16	7,6	7,1%	0,0%	44,4%	0,0%	4,9%	0,0%	0,0%	31,7%	0,0%	11,9%	
17	7,2	6,4%	0,0%	53,5%	0,0%	2,1%	0,0%	0,0%	38,0%	0,0%	0,0%	
18	6,7	3,0%	0,0%	67,9%	0,0%	0,0%	0,0%	0,0%	29,1%	0,0%	0,0%	
19	6,1	2,5%	0,0%	83,5%	0,0%	0,0%	0,0%	0,0%	14,0%	0,0%	0,0%	
20	5,5	0,0%	0,0%	100,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	

(Panel b.) Adjusted-Hurst scaling method

Port. #	RETURN / RISK	COMMODITIES	EMERGING MARKETS EQUITY	EMERGING MARKETS EMBI	DEVELOPED MARKETS EQUITY	US.TREAS	US.CORP	US.MRTG	GER.TRES	JAP.TREAS	UK.TREAS	
		0,482	**0,585**	**0,589**	**0,512**	**0,517**	**0,499**	**0,495**	**0,513**	**0,501**	**0,526**	*←Adj. H exponent*
		1,230	**0,406**	**2,756**	**0,748**	**5,375**	**6,014**	**9,334**	**6,538**	**3,736**	**6,094**	*←Return/Risk*
1	8,5	1,3%	0,3%	0,0%	1,1%	11,9%	0,0%	0,2%	13,2%	68,7%	3,2%	
2	10,4	1,4%	0,3%	0,0%	1,1%	3,9%	3,2%	9,0%	13,2%	58,9%	8,9%	
3	11,5	1,9%	0,3%	0,0%	1,2%	0,0%	5,2%	16,6%	12,5%	47,9%	14,5%	
4	11,9	2,6%	0,2%	0,1%	1,3%	0,0%	4,7%	23,6%	11,5%	35,9%	20,2%	
5	12,0	3,1%	0,2%	0,2%	1,4%	0,0%	4,9%	28,8%	11,7%	25,1%	24,5%	
6	11,9	3,6%	0,2%	0,3%	1,6%	0,0%	5,5%	33,3%	12,5%	14,8%	28,3%	
7	11,7	4,6%	0,0%	0,9%	1,3%	0,0%	2,1%	43,9%	2,4%	8,3%	36,5%	
8	11,2	5,9%	0,0%	2,2%	0,0%	0,0%	0,0%	50,4%	7,1%	3,7%	30,7%	
9	10,3	7,4%	0,0%	3,7%	0,0%	0,0%	0,0%	53,8%	17,3%	0,0%	18,0%	
10	9,3	9,3%	0,0%	5,3%	0,0%	0,0%	0,0%	55,4%	29,5%	0,0%	0,4%	
11	8,3	11,4%	0,0%	8,2%	0,0%	0,0%	0,0%	38,2%	42,3%	0,0%	0,0%	
12	7,5	13,5%	0,0%	11,0%	0,0%	0,0%	0,0%	20,5%	55,0%	0,0%	0,0%	
13	6,8	15,6%	0,0%	13,8%	0,0%	1,9%	0,0%	2,0%	66,7%	0,0%	0,0%	
14	6,0	11,5%	0,0%	23,1%	0,0%	0,0%	0,0%	0,0%	65,4%	0,0%	0,0%	
15	5,2	9,9%	0,0%	34,2%	0,0%	0,0%	0,0%	0,0%	55,9%	0,0%	0,0%	
16	4,4	11,4%	0,0%	45,9%	0,0%	0,0%	0,0%	0,0%	42,7%	0,0%	0,0%	
17	3,8	12,9%	0,0%	57,7%	0,0%	0,0%	0,0%	0,0%	29,4%	0,0%	0,0%	
18	3,4	14,4%	0,0%	69,4%	0,0%	0,0%	0,0%	0,0%	16,2%	0,0%	0,0%	
19	3,1	15,9%	0,0%	81,1%	0,0%	0,0%	0,0%	0,0%	3,0%	0,0%	0,0%	
20	2,8	0,0%	0,0%	100,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	

Source: authors' calculations.

Table 5 and Figure 8 present a summary of the weights assigned by both methods to the efficient frontier.

Table 5
1-year and 10-year horizon efficient frontier's weights (summary)

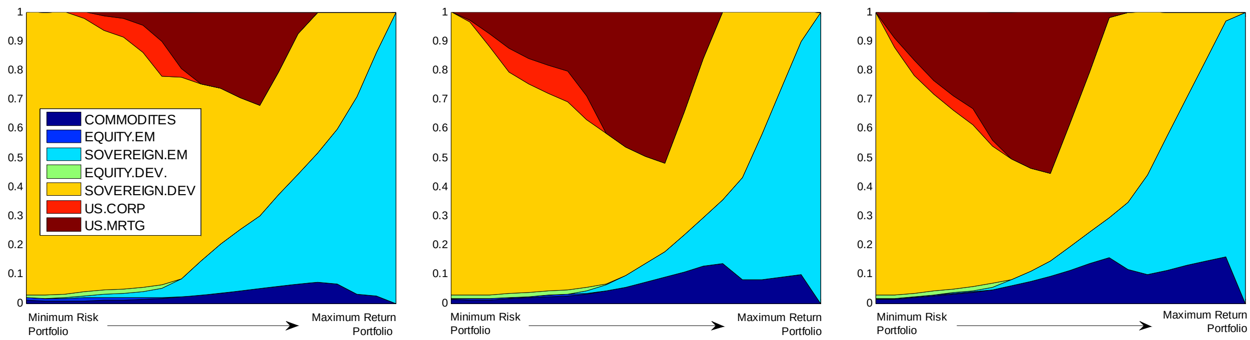
		COMMODITIES	EMERGING MARKETS EQUITY	EMERGING MARKETS EMBI	DEVELOPED MARKETS EQUITY	US.TREAS	US.CORP	US.MRTG	GER.TRES	JAP.TREAS	UK.TREAS
Adj. H exponent →		*0,482*	*0,585*	*0,589*	*0,512*	*0,517*	*0,499*	*0,495*	*0,513*	*0,501*	*0,526*
H=0,5 (*n* -years)	Mean	2,9%	0,3%	25,4%	0,5%	4,6%	1,9%	8,9%	16,8%	11,2%	27,4%
	Max	7,1%	1,0%	100,0%	1,4%	15,0%	12,1%	31,9%	38,0%	58,4%	67,4%
	Min	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%
H=Hadj (1-years)	Mean	5,6%	0,2%	20,2%	0,6%	1,6%	2,5%	17,3%	24,0%	14,3%	13,8%
	Max	13,5%	0,6%	100,0%	1,7%	11,9%	10,6%	51,8%	62,2%	66,0%	47,9%
	Min	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%
H=Hadj (10-years)	Mean	7,9%	0,1%	22,9%	0,4%	0,9%	1,3%	18,8%	25,4%	13,2%	9,3%
	Max	15,9%	0,3%	100,0%	1,6%	11,9%	5,5%	55,4%	66,7%	68,7%	36,5%
	Min	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%

Source: authors' calculations.

Figure 8

Square-root-of-time and Adjusted-Hurst methods for 1-year and 10-year weights

H=0.5 (n-year) *$H=\breve{H}$ (1-year)* *$H=\breve{H}$ (10-year)*



Source: authors' calculations.

5. Final remarks

Most effort has been given to financial assets' returns short-term dependence. In this sense many models are readily available to improve the estimation of the variance and to a lesser degree covariance inputs for portfolio construction.

Less emphasis has been given to long-term dependence of returns. Akin to financial literature this document shows that (i) significant long-term dependence is common in assets' returns time-series; (ii) significant persistence is prevalent for emerging fixed income markets, and fairly frequent for emerging equity markets –mainly the less liquid ones; (iii) independence is representative of developed fixed income and equity markets, and somewhat recurrent for liquid emerging equity markets; (iv) energy markets exhibit significant antipersistence.

Interestingly, this document's support for prior evidence includes data from the most recent and severe episode of widespread financial disruption. Divergence with documented literature is circumscribed to authors' findings of recurrent antipersistence for developed equity markets, as well as a few liquid emerging markets.

This document's long-term dependence assessment relies on rescaled range analysis *(R/S)*, a popular and robust methodology designed for Geophysics but extensively used in financial literature. Well-known issues of *R/S* such as the optimal minimum and maximum size of periods were surmounted *vis-à-vis* some previous studies, resulting in reduced estimators' standard errors and minimal interference of short-term serial dependence in the results.

Ahead of *R/S* financial literature, authors used the spread between estimated Hurst exponent ($\widehat{H}$) and the expected Hurst exponent for independent and finite time-series ($\dot{H}$) to estimate an adjusted Hurst exponent ($\breve{H}$). Under a generalized version of the conventional volatility and covariance scaling procedure, authors suggest using this adjusted measure of long-term dependence for practical purposes, where long-term mean-variance portfolio optimization is a natural choice to begin with.

Comparing efficient portfolio weights resulting from customary mean-variance optimization (e.g. independency assumption reliant) and the suggested enhanced procedure shows that the former tends to overweight persistent risk factors. Once long-term dependence is considered *via* the proposed covariance scaling procedure, the return per unit of risk of persistent (antipersistent) risk factors is adjusted downwards (upwards), decreasing (increasing) the weight of high (low) persistence risk factors as the investment horizon increases. Results provide evidence of the significance of weight differences for 1-year and 10-year investment horizons and of how these differences reveal that adjusted efficient frontiers are less optimistic (e.g. there is a lower level of return for each level of risk) than conventional ones.

Resulting less optimistic efficient frontiers and their corresponding weights also reveals that long-term dependence recognition conveys various practical advantages, especially for long-term institutional investors, such as central banks, pension funds and sovereign wealth managers. First, because the proposed scaling procedure exposes concealed riskiness resulting from persistence, extreme relative return/risk ratios differences due to inappropriate risk scaling are moderated, avoiding to some extent excessive risk taking in long-term portfolios and mitigating the presence of extreme portfolio weights.

Second, evidence of significant persistence in small and illiquid capital markets provides proof of masked risks within their securities. Such underestimation of local instruments' long-term risk could explain two well-known facts of those capital markets: (i) the tendency to hold a disproportionate level of investments within the domestic market or "home bias", and (ii) the reluctance to hold foreign currency-denominated assets. Recognizing long-term dependence would make local –persistent- instruments from small and illiquid markets less attractive within the mean-variance asset allocation framework, and developed markets' –independent or antipersistent- instruments more attractive.

Given these insights the authors are currently considering three research topics. Firstly, to study the contribution of individual risk factors to portfolio's persistence. Initial results herein presented show that persistence at the portfolio level can be significantly higher than the weighted persistence of individual assets, especially for small and illiquid markets, thereby reinforcing the international diversification case.

Secondly, akin to upside and downside risk concepts, authors also envision a methodology capable of differentiating upside from downside persistence. This is a key issue because persistence may be an asset's desirable (undesirable) feature if its price is expected to rise (fall) in the future (e.g. a persistent bond may be attractive on the verge of monetary expansion). In the meanwhile authors suggest considering market's environment and investors' views in order to decide the convenience of underweighting persistent risk factors. Alternatively, including optimization constraints such as a threshold for maximum drawdown (Reveiz and León, 2010) may capture investor's natural inclination (reluctance) to hold upside (downside) persistent risk factors.

Finally, because Black-Litterman portfolio optimization is heavily reliant on the serial long-term independence assumption *via* traditional volatility scaling and the starting global CAPM equilibrium, authors' agenda also includes designing long-term dependence adjustments to this celebrated approach.

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7. Appendix

A. For a time series of N returns, having k independent (non overlapping[9]) windows or samples of size n, divide the original series in such way that $n \times k = N$.

B. Estimate the arithmetic mean of each k-segment $(\hat{\mu}_k)$ of size n .

C. Obtain the difference between each i-return and the mean of each k segment $(\hat{\mu}_k)$.

$$Y_{i,k} = x_{i,k} - \hat{\mu}_k$$

D. Calculate accumulative differences for each k segment.

$$D_{i,k} = \sum_{i=1}^{n} Y_{i,k}$$

E. Calculate range ($R_{n,k}$) of the $D_{i,k}$ series.

$$R_{n,k} = max\left(D_{1,k}, \dots, D_{i,k}, \dots D_{n,k}\right) - \ min\left(D_{1,k}, \dots, D_{i,k}, \dots D_{n,k}\right)$$

F. Estimate standard deviation for each k segment $(S_{n,k})$.

$$S_{n,k} = \sqrt{\frac{1}{(n-1)} \sum_{i=1}^{n} \left(x_{i,k} - \hat{\mu}_k\right)^2}$$

G. Calculate rescaled range for each segment k.

$$(R/S)_{n,k} = {R_{n,k}}/{S_{n,k}}$$

H. Calculate average rescaled range for k segments of size n.

$$(R/S)_n = \frac{1}{k} \sum_{i=1}^{k} (R/S)_{n,k}$$

$(R/S)_n$ corresponds to average standardized distance covered per unit of time n.

The previous procedure must be done for different values of k, where $k_j = n_{min} \dots\ n_{max}$, and where n_{min} y n_{max} corresponds to the minimum and maximum of the chosen window to calculate the rescaled range. Thus, we have j values of $(R/S)_n$, where $n_j = \frac{N}{k_j}$.

Finally, using n and $(R/S)_n$ values we estimate the ordinary least squares regression proposed by Mandelbrot and Wallis (1969a y 1969b), where H corresponds to the estimated Hurst exponent:

$$Log(R/S)_n = Log(c) + HLog(n)$$

[9] For a discussion regarding the use of overlapping and non-overlapping segments, please refer to Nawrocki (1995) and Ellis (2007).