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ESTIMATIONS OF THE NATURAL RATE OF INTEREST IN COLOMBIA*

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ABSTRACT. Three methodologies to estimate the natural interest rate, NIR, are implemented for the Colombian economy. Two methods are statistical filters and the third involves some economic theory. The first method is based on unobserved components decomposition of the real interest rate and explores the statistical characteristics of the data. The second is a multivariate version of the Hodrick-Prescott filter augmented by an economic relationship, HPMV. The NIR in both cases is defined as the trend component of the market real interest rate; then, the NIR may be considered as a long-run real interest rate anchor for monetary policy. The third method consists in estimating a semi-structural model for a small open economy. In this case the NIR is defined as the interest rate that does not affect the output dynamics in the short run and assures output and inflation convergence to their long run equilibriums. This implies that the NIR is a medium-run anchor for monetary policy. Three features are observed in the dynamics of the NIR estimates for the period 2000-2009. The first part of the sample (2000-2003) shows a downward trend, followed by a period of stabilization and upward trend (2004-2008) and at the end of the sample the NIR start decreasing again. The NIR in the last quarter of the sample, 2009Q2, is around 3.1 in average.

Key words and phrases. Natural Rate of Interest, Unobserved Components Models, Hodrick-Prescott Filter, Semi-structural models.

JEL clasification. C13, C32, E43, E52.

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1. Introduction

In monetary policy regimes, where the nominal short run interest rate is used as the policy instrument, as is the case of Colombia since 2001, the natural interest rate, NIR, plays an important role in order to determine the stance of monetary policy. Thus, the gap between the instrument rate of the Central Bank and the NIR can be a useful guideline for the position of the monetary policy and can also be helpful to make policy decisions (Laubach and Williams [2001]).

There are several definitions of the natural or neutral interest rate, however the standard definition states that the natural interest rate is the short run interest rate which makes the output to converge to its potential keeping inflation stable (Bomfim [1997]).

In the literature, different approaches have been used to estimate the NIR. From simple statistical methodologies to structural economic models. Basdenvant et al. [2004] used a multivariate Hodrick-Prescott filter, HPMV, to estimate the NIR in New Zealand; Crespo Cuaresma et al. [2003] used an unobserved components models, UCM, to estimate the NIR for the Euro area. More complex methodologies such as Stochastic Dynamic General Equilibrium models were used by Neiss and Nelson [2001] for the UK and Giammarioli and Valla [2003] for the Euro area. Also, some semi-structural or more parsimonious models have been used by Laubach and Williams [2001] for USA. They jointly estimate the trend growth of the economy, the NIR and the potential output by the Kalman filter.

There are several estimations of the natural rate of interest for Latin American economies. In particular, España [2008] used the methodology of Laubach and Williams [2001] for the Uruguayan economy; in the case of Peru, Castillo et al. [2006] estimated the NIR by Kalman filter using a semi-structural model for a small open economy. Calderon and Gallego [2002] estimated the neutral interest rate using two approaches for the Chilean economy; first, they considered theoretical rates that would prevail under equilibrium conditions in a closed and open economy and second, they used rates derived from expectations of the interest rate from the monetary authority and the financial market. Finally, for Venezuela, Cartaya et al. [2007] estimated the NIR first, from the marginal productivity of capital derived from a production function and second, by the Kalman filter using a small system of equations that includes a relation between the output gap and the interest rate gap and the dynamics of the potential output and the NIR.

In the Colombian case, Echavarría et al. [2006] estimate the NIR based on the work of Laubach and Williams [2001], using quarterly data for the period 1982 Q1 to 2005 Q4. It is a closed economy model since it does not incorporate relationships of the interest rates across countries. Although the model includes some external variables such as the terms of trade and foreign growth.

Another study is the Transmission mechanism model which is the model currently used by the central bank of Colombia for monetary policy simulation and long run forecasting (Gómez et al. [2002]). It assumes that the NIR is constant at 4% for the last part of the sample. Additionally, for this model a neutral real interest rate is estimated. This neutral rate is defined as the transition rate that converges to the long run or stationary state. This path is used to obtain an estimation of the interest rate gap.

In this paper we obtain estimates of the time varying natural interest rate using three methodologies. Two statistical methods, based on the dynamic properties of the data, and a semi-structural model for a small open economy.

The statistical methods are based on UCM and HPMV models. For both methods, the NIR is estimated as the trend component of the real interest rate. First, two UCM versions are estimated, univariate and multivariate models. Second, a Multivariate Hodrick-Prescott filter, proposed by Laxton and Tetlow [1992], is estimated. This method adds an economic relationship to the Hodrick-Prescott optimization problem in order to obtain the trend component.

On the other hand, a semi-structural model for a small open economy is considered. This methodology simultaneously estimate the output gap, potential output, core inflation and the NIR using the Kalman Filter. The model is based on the work of Castillo et al. [2006]; with the advantage that inflation expectations are estimated within the system and the parameters estimates obtained by bayesian techniques. The parameters estimation were carried out with the methodology described in Bonaldi et al. [2010] using their FORTRAN95 procedures[1].

The remainder of the paper is structured as follows. Section 2 describes the methodologies used to estimate the NIR. The description of the data used in this analysis is presented in Section 3. Section 4 shows the estimates of the NIR. Finally, Section 5 concludes.

2. Methodologies of Estimation of the NIR

This section briefly describes three methodologies that are used to estimate the Colombian NIR. The first two are based on statistical methods while the last one uses a semi-structural model for a small open economy. The statistical methods, unobserved components model and an augmented Hodrick-Prescott filter, extract the long-run trend of the real interest rate as a measure of the natural interest rate. On the other hand, the semi-structural model considers the relations between the natural interest rate and different macroeconomic variables according to the economic theory. In the latter approach, the NIR is defined as the interest rate that does not affect the output dynamics in the short run and ensures output and inflation convergence to their long run equilibriums.

2.1. **Unobserved Components Models.** Unobserved components models decompose a time series into several components such as trend, season, cycle and irregular disturbance. These models have been intensively used in applied economic research and successfully applied in business cycle analysis. They are also useful in short-term monitoring of macroeconomic variables. Compared with other filtering procedures (Hodrick -

[1]We gratefully acknowledge the authors for providing us their full codes

Prescott, X-11 and X-12), the unobserved components models offer some advantages. It provides statistical tests, prediction algorithms, modeling of seasonality and introduction of additional features such as other explanatory variables, interventions and cyclical components.

The unobserved components models have been used in different economic applications, for estimating the natural level of the labor supply (Bull and Frydman [1983]), for modeling credibility of the monetary authority (Weber [1991]), for analyzing the GDP (Luginbuhl and Vos [1999], Morley et al. [2003]), the Purchasing Power Parity (PPP) (Kleijn and van Dijk [2001]), consumption (Elwood [1998]), unemployment (Chung and Harvey [2000], Berger and Everaert [2009]), for modeling tax revenues (Koopman and Ooms [2003]), cycles (Chambers and McGarry [2002]) and for analyzing financial series (Cowan and Joutz [2006]), among others.

2.1.1. *Univariate Model.* In this section we introduce the unobserved component models (UCM) developed by Harvey [1989]. Let y_t be the observed time series which is decomposed into several components in the following way:

$$y_t = \mu_t + \gamma_t + \varphi_t + \epsilon_t, \quad \epsilon_t \sim NID(0, \sigma^2_\epsilon), \quad t = 1, \dots, N \tag{2.1}$$

where $\mu_t, \gamma_t, \varphi_t$ and ϵ_t represent the trend, seasonal, cyclical and irregular components, respectively.

The trend is modeled by a linear stochastic process that may include a slope term. The seasonal components can be modeled by a linear stochastic process, trigonometric functions or deterministic components. The cycle is based on stochastic trigonometric functions.

The specification of the UCM depends on which components are included in the model and how they are modeled. Thus, the simplest form of UCM, the "local level model", is obtained from equation (2.1), with no cyclical and no seasonal components and by specifying the trend as a random walk process

$$y_t = \mu_t + \epsilon_t, \qquad \epsilon_t \sim NID(0, \sigma^2_\epsilon), \quad t = 1, \dots, N \tag{2.2}$$

$$\mu_{t+1} = \mu_t + \eta_{t+1}, \qquad \eta_t \sim NID(0, \sigma^2_\eta) \tag{2.3}$$

The "local linear trend model" is obtained from the previous model by adding a slope term β_t, which also follows a random walk process

$$y_t = \mu_t + \epsilon_t, \qquad \epsilon_t \sim NID(0, \sigma^2_\epsilon) \tag{2.4}$$

$$\mu_{t+1} = \mu_t + \beta_t + \eta_{t+1}, \qquad \eta_t \sim NID(0, \sigma^2_\eta), \quad t = 1, \dots, N \tag{2.5}$$

$$\beta_{t+1} = \beta_t + \zeta_{t+1}, \qquad \zeta_t \sim NID(0, \sigma^2_\zeta) \tag{2.6}$$

where the trend and slope disturbances, η_t and ζ_t, are mutually uncorrelated Gaussian sequences with zero mean and variances σ_η^2 and σ_ζ^2. If σ_ζ^2 is zero then the trend μ_t follows a random walk process plus drift. Moreover, if $\sigma_\eta^2 = \sigma_\zeta^2 = 0$ then μ_t is a deterministic linear trend. A "smooth trend model" or an integrated random walk process is obtained when $\sigma_\eta^2 = 0$.

To take into account the seasonal variation in y_t, the seasonal component can be specified by a deterministic or stochastic component. The deterministic seasonal component satisfies the property that the seasonal coefficients sum zero within a year. This ensures that this component is not interpreted as a trend. In this case, the deterministic seasonal component is given by

$$\gamma_t = \sum_{j=1}^{s-1} \tilde{\gamma}_j z_{jt} \tag{2.7}$$

where s is the number of seasons and z_{jt} is a dummy variable that indicates if observation t belongs to the $j-$season and $\tilde{\gamma}_j$ for $j = 1, \ldots, s$ are the respective coefficients.

Finally, an alternative way of modeling seasonality is given by

$$\gamma_t = \sum_{j=1}^{[s/2]} (\alpha_j \cos \lambda_j t + \beta_j \sin \lambda_j t) \tag{2.8}$$

where $\lambda_j = 2\pi j/s, \ \ j = 1, \ldots, [s/2]$, and $[.]$ denotes rounding down to the nearest integer.

Time series are often subject to certain economic fluctuations that can be interpreted as business cycles. This can be implemented in UCM by including a stochastic cycle (φ_t) of the form

$$y_t = \mu_t + \varphi_t + \epsilon_t, \quad \epsilon_t \sim NID(0, \sigma_\epsilon^2), \quad t = 1, \ldots, N \tag{2.9}$$

$$\begin{pmatrix} \varphi_{t+1} \\ \varphi_{t+1}^* \end{pmatrix} = \rho \begin{pmatrix} \cos \lambda & \sin \lambda \\ -\sin \lambda & \cos \lambda \end{pmatrix} \begin{pmatrix} \varphi_t \\ \varphi_t^* \end{pmatrix} + \begin{pmatrix} \kappa_{t+1} \\ \kappa_{t+1}^* \end{pmatrix}, \ \ 0 \le \rho < 1 \tag{2.10}$$

where κ_t and κ_t^* are white noise disturbances mutually uncorrelated with common variance σ_κ^2, the trend μ_t can be specified by (2.3) or (2.5) and (2.6), ρ is a damping factor, λ is the frequency in radians corresponding to a period $2\pi/\lambda$ such that $0 < \lambda < \pi$. For more details see Harvey [1981].

2.1.2. *Multivariate Model.* The multivariate version of unobserved components models extends the results of section 2.1.1 for a vector of variables. Let $\mathbf{y}_t$ be a vector of k observed variables. Then, the model can be written as the following additive form

$$\mathbf{y}_t = \boldsymbol{\mu}_t + \boldsymbol{\gamma}_t + \boldsymbol{\varphi}_t + \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim NID(\mathbf{0}, \Sigma_\epsilon), \quad t = 1, \ldots, N \tag{2.11}$$

where $\boldsymbol{\mu}_t$, $\boldsymbol{\gamma}_t$, $\boldsymbol{\varphi}_t$ and $\boldsymbol{\epsilon}_t$ are $k \times 1$ vectors that correspond to the multivariate trend, seasonal, cycle and irregular components, respectively. A simple model includes a multivariate trend and cycle components and is given by

$$\mathbf{y}_t = \boldsymbol{\mu}_t + \boldsymbol{\varphi}_t + \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim NID(\mathbf{0}, \Sigma_\epsilon), \quad t = 1, \ldots, N \tag{2.12}$$

where $\boldsymbol{\mu}_t$ can be expressed either in a multivariate local level model or in a local linear trend model. The former has the following form

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\mu}_t + \boldsymbol{\eta}_{t+1}, \qquad \boldsymbol{\eta}_t \sim NID(\mathbf{0}, \Sigma_\eta), \tag{2.13}$$

And the multivariate local linear trend is given by

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\mu}_t + \boldsymbol{\beta}_t + \boldsymbol{\eta}_{t+1}, \qquad \boldsymbol{\eta}_t \sim NID(\mathbf{0}, \Sigma_\eta), \tag{2.14}$$

$$\boldsymbol{\beta}_{t+1} = \boldsymbol{\beta}_t + \boldsymbol{\zeta}_{t+1}, \qquad \boldsymbol{\zeta}_t \sim NID(\mathbf{0}, \Sigma_\zeta), \tag{2.15}$$

The equation of the cycle in both models is given by

$$\begin{pmatrix} \boldsymbol{\varphi}_{t+1} \\ \boldsymbol{\varphi}^*_{t+1} \end{pmatrix} = \rho \left[\begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix} \otimes I_k \right] \begin{pmatrix} \boldsymbol{\varphi}_t \\ \boldsymbol{\varphi}^*_t \end{pmatrix} + \begin{pmatrix} \boldsymbol{\kappa}_{t+1} \\ \boldsymbol{\kappa}^*_{t+1} \end{pmatrix}, \quad V \left(\begin{array}{c} \boldsymbol{\kappa}_t \\ \boldsymbol{\kappa}^*_t \end{array} \right) = I_2 \otimes \Sigma_\kappa \tag{2.16}$$

where the cyclical frequency λ and the cycle damping factor ρ, $0 < \rho < 1$, are assumed to be equal for all variables. The disturbances, $\boldsymbol{\kappa}_t$ and $\boldsymbol{\kappa}^*_t$ are two orthogonal white noise processes.

In a simple multivariate local level model, when the rank of the Σ_η is $k^* < k$, the model has k^* common levels or common trends, $\boldsymbol{\mu}^*_t$. Then, equations (2.12) and (2.13), with no cycle component, becomes

$$\mathbf{y}_t = \Theta^* \boldsymbol{\mu}^*_t + \boldsymbol{\mu}_0 + \boldsymbol{\epsilon}_t, \qquad \boldsymbol{\epsilon}_t \sim NID(\mathbf{0}, \Sigma_\epsilon), \tag{2.17}$$

$$\boldsymbol{\mu}^*_{t+1} = \boldsymbol{\mu}^*_t + \boldsymbol{\eta}^*_{t+1}, \qquad \boldsymbol{\eta}^*_t \sim NID(\mathbf{0}, \Sigma_{\eta^*}) \tag{2.18}$$

where $\boldsymbol{\mu}^*_t$ is a $k^* \times 1$ vector of *common trends*, Θ^* is an $k \times k^*$ matrix of factor loadings, $\boldsymbol{\mu}_0$ is a $k-$dimensional vector which has zeros for the first k^* elements and the remaining elements are unconstrained ($\bar{\boldsymbol{\mu}}$). The presence of common trends implies cointegration (Harvey [1989]). In the local level model, there are $r = k - k^*$ cointegration vectors. Equation (2.17) can also be expressed as

$$\mathbf{y}_{1,t} = \boldsymbol{\mu}_t^* + \boldsymbol{\epsilon}_{1t} \tag{2.19}$$
$$\mathbf{y}_{2,t} = \Theta^* \boldsymbol{\mu}_t^* + \bar{\boldsymbol{\mu}} + \boldsymbol{\epsilon}_{2t} \tag{2.20}$$

where $\mathbf{y}_t$ is partitioned into a $k^* \times 1$ vector $\mathbf{y}_{1t}$ and an $r \times 1$ vector $\mathbf{y}_{2t}$, $\boldsymbol{\epsilon}_t$ is partitioned in a similar way. The first set contains the common trends and the second set of equations consists of cointegrating relationships.

The model described in (2.17) and (2.18) can be written in the form of seemingly unrelated time series equations (SUTSE), this representation is a multivariate generalization of standard structural time series models and common components restrictions, such as common trends, common cycles and common seasonalities. For the previous model the SUTSE representation is given by

$$\mathbf{y}_t = \boldsymbol{\mu}_t + \boldsymbol{\epsilon}_t, \qquad \boldsymbol{\epsilon}_t \sim NID(\mathbf{0}, \Sigma_\epsilon) \tag{2.21}$$
$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\mu}_t + \boldsymbol{\eta}_{t+1}, \qquad \boldsymbol{\eta}_t \sim NID(\mathbf{0}, \Sigma_\eta) \tag{2.22}$$

Where $\boldsymbol{\mu}_t = \Theta^* \boldsymbol{\mu}_t^* + \boldsymbol{\mu}_0$, $\boldsymbol{\eta}_t = \Theta^* \boldsymbol{\eta}_t^*$ and $\Sigma_{\boldsymbol{\eta}} = \Theta^* \Sigma_{\boldsymbol{\eta}^*} \Theta^{*'}$ is a singular matrix of rank k^*.

The multivariate local linear trend model with cycles and common levels is given by

$$\mathbf{y}_t = \Theta^* \boldsymbol{\mu}_t^* + \boldsymbol{\mu}_0 + \boldsymbol{\varphi}_{\boldsymbol{t}} + \boldsymbol{\epsilon}_t, \qquad \boldsymbol{\epsilon}_t \sim NID(\mathbf{0}, \Sigma_\epsilon), \tag{2.23}$$
$$\boldsymbol{\mu}_{t+1}^* = \boldsymbol{\mu}_t^* + \boldsymbol{\beta}_t^* + \boldsymbol{\eta}_{t+1}^*, \qquad \boldsymbol{\eta}_t^* \sim NID(\mathbf{0}, \Sigma_{\eta^*}), \tag{2.24}$$
$$\boldsymbol{\beta}_{t+1}^* = \boldsymbol{\beta}_t^* + \boldsymbol{\zeta}_{t+1}^*, \qquad \boldsymbol{\zeta}_t^* \sim NID(\mathbf{0}, \Sigma_\zeta{}^*), \tag{2.25}$$

The SUTSE representation of this model is given in the equations (2.12), (2.14), (2.15) and (2.16) when $\boldsymbol{\mu}_t = \Theta^* \boldsymbol{\mu}_t^* + \boldsymbol{\mu}_0$, $\boldsymbol{\beta}_t = \Theta^* \boldsymbol{\beta}_t^*$, $\boldsymbol{\zeta}_t = \Theta^* \boldsymbol{\zeta}_t^*$, $\Sigma_{\boldsymbol{\zeta}} = \Theta^* \Sigma_{\boldsymbol{\zeta}^*} \Theta^{*'}$, $\boldsymbol{\eta}_t = \Theta^* \boldsymbol{\eta}_t^*$ and $\Sigma_{\boldsymbol{\eta}} = \Theta^* \Sigma_{\boldsymbol{\eta}^*} \Theta^{*'}$. Σ_η and Σ_ζ are singular matrices with rank k^*.

Then, SUTSE models allow to identify common factors through covariance matrices of the disturbances $(\Sigma_\epsilon, \Sigma_{\eta^*}, \Sigma_{\zeta^*}, \ldots)$. An incomplete rank of any of these matrices implies a common component restriction. For example, if the covariance matrix of the trend disturbance has incomplete rank, then there are common trends. Since SUTSE models may include cointegration relationships, they allow for economic interpretations and can also provide more efficient forecasts (Mazzi et al. [2005]). Another advantage of these models is that they provide a useful framework for temporal disaggregation (Moauro and Savio [2005]).

As shown in Appendix C, both, univariate and multivariate unobserved component models can be written in a state space form and estimated by Maximum Likelihood using the Kalman filter.

2.2. **Multivariate Hodrick - Prescott Filter.** This alternative of estimating unobserved components, known as HPMV, was developed by Laxton and Tetlow [1992] to estimate potential output in Canada, and has also been used for estimating potential output in New Zealand (Conway and Hunt [1997]) and estimating the NAIRU for the OECD countries (OECD [1999]). This methodology is based on the Hodrick- Prescott filter, however, an additional economic equation related to the unobserved trend component is considered in the optimization problem. In this way, the minimization problem also depends on the fit of the economic relationship. The HPMV is obtain as the solution of the following problem

$$\min \sum \frac{1}{\sigma_0^2} \left(y_t - y_t^*\right)^2 + \frac{1}{\sigma_1^2} \left(\Delta\Delta y_t^*\right)^2 + \frac{1}{\sigma_2^2} \xi^2 \tag{2.26}$$

where σ_0^2, σ_1^2, σ_2^2 are the variances of the cyclical fluctuations $(y_t - y_t^*)$, the growth rate of the trend $(\Delta\Delta y_t^*)$ and the errors of the economic relationship (ξ_t), respectively. Then, the smaller σ_2^2, the higher λ_2, the more importance is given to the information added by the economic relationship.

In this study, two economic equations are considered for the estimation of the NIR using this methodology.

First, an IS curve, which relates interest rates and income is considered. This equation represents the equilibrium of the market of goods and services. The relationship is given by

$$\widetilde{y}_t = \alpha_0 + \alpha_1 \widetilde{y}_{t-1} + \alpha_2 \left(r_{t-1} - \bar{r}_{t-1}\right) + \alpha_3 \tilde{q}_{t-1} + \xi_t \tag{2.27}$$

where $\widetilde{y}_t$ is the output gap, $\tilde{q}_t$ is the real exchange rate gap, r_t is the real interest rate and $\bar{r}_t$ is the NIR.

Second, the following Taylor policy rule is also considered

$$i_t^{ON} = \rho i_{t-1}^{ON} + (1-\rho)\left[(\bar{r}_t + E_t \pi_{t+s} - \gamma_t) + \alpha_1(\pi_{t+s} - \bar{\pi}_{t+s}) + \alpha_2 \widetilde{y}_t\right] + \xi_t \tag{2.28}$$

where i_t^{ON} is the nominal overnight rate, $\bar{r}_t$ is the NIR, $E_t \pi_{t+s}$ is the inflation expectation for $s = 12$ periods ahead, defined as the weighted average of past and future inflation, γ_t is the gap between the unobserved nominal natural overnight rate and nominal natural market rate (obtained from the 90-day deposit interest rate), $\gamma_t = \bar{i}_t^{90TD} - \bar{i}_t^{ON}$, and $\bar{\pi}_t$ is the inflation target.

There are two alternative methods to estimate the HPMV filter. Using optimization methods to solve the minimization problem and using the Kalman filter to estimate the respective state-space representation.

2.2.1. *The optimization.* To start the minimization problem, a proxy of the unobserved trend, usually obtained from the standard HP filter, is used to estimate the economic equation. Then, the residuals from this equation are plugged into the following minimization problem (2.26), with $\lambda_1 = \frac{\sigma_0^2}{\sigma_1^2}$ and $\lambda_2 = \frac{\sigma_0^2}{\sigma_2^2}$,

$$\min \sum \left((r_t - \bar{r}_t)^2 + \lambda_1 (\bar{r}_{t+1} - 2\bar{r}_t + -\bar{r}_{t-1})^2 + \lambda_2 (z_t - f(\bar{r}_t))^2 \right) \tag{2.29}$$

Where z_t is the dependant variable of the economic relationship and $f(\bar{r}_t)$ is the linear function of the economic model.

As shown in Razzak and Dennis [1999], the first order conditions with respect to $\bar{r}_t$ for $t = 1, \ldots, T$, implies that

$$\boldsymbol{c} = \lambda_1 F \bar{\boldsymbol{r}} + \lambda_2 \boldsymbol{\xi}^* \tag{2.30}$$

where $\boldsymbol{c}$ is a $T \times 1$ vector with $c_t = r_t - \bar{r}_t$, $\boldsymbol{\xi}^*$ is a vector of derivatives of $\sum (z_t - f(\bar{r}_t))^2$ with respect to $\bar{r}_t$, F is a $T \times T$ matrix of the following form

$$F = \begin{bmatrix} 1 & -2 & 1 & 0 & & & & \cdots & 0 \\ -2 & 5 & -4 & 1 & 0 & & & \cdots & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & & \cdots & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & \cdots & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & \cdots & & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & \cdots & & & 0 & 1 & -4 & 5 & -2 \\ 0 & \cdots & & & & 0 & 1 & -2 & 1 \end{bmatrix} \tag{2.31}$$

For the IS curve, $\boldsymbol{\xi}^* = \alpha_2 [\xi_2, \xi_3, \ldots, \xi_T]'$, in this case F is a $T-1 \times T-1$ matrix. In the monetary policy rule exercise, $\boldsymbol{\xi}^* = -(1-\rho) [\xi_1, \xi_2, \ldots, \xi_{T-1}, \xi_T]'$.

Under this methodology the real interest rate is decomposed into the unobserved trend and cyclical residual components, $\boldsymbol{r} = \bar{\boldsymbol{r}} + \boldsymbol{c}$. Replacing $\boldsymbol{c}$ by (2.30) in the previous expression and solving for the trend component, $\bar{\boldsymbol{r}}$, the following result is obtained [2]

$$\bar{\boldsymbol{r}} = (I + \lambda_1 F)^{-1} (\boldsymbol{r} - \lambda_2 \boldsymbol{\xi}^*) \tag{2.32}$$

Once the unobserved $\bar{\boldsymbol{r}}$ is obtained, an optimization algorithm is used to estimate the $\lambda's$. Then, the economic relationship is estimated with the new estimate of $\bar{\boldsymbol{r}}$ using NLS. Several iterations of this three-step procedure are performed until convergence is reached.

[2] See Reeves et al. [1996] and Conway and Hunt [1997] for details.

2.2.2. *The Kalman filter.* The HPMV filter can be rewritten as a state space model with some restrictions imposed over the variances of the three components of the minimization problem. These restrictions are implemented in order to produce a balance among smoothness, bias and the fit of the economic relationship. Estimates of the trend component and the parameters are obtained by maximum likelihood methodology using Kalman filter.

The state space representation of the HPMV minimization problem for the first economic relationship is given by:

Measurement equation

$$\boldsymbol{y}_t = d\boldsymbol{X}_t + z\boldsymbol{A}_t + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \overset{iid}{\sim} N\left(\mathbf{0}, GG'\right) \tag{2.33}$$

Transition equation

$$\boldsymbol{A}_t = T\boldsymbol{A}_{t-1} + \boldsymbol{\nu}_t, \qquad \boldsymbol{\nu}_t \overset{iid}{\sim} N\left(\mathbf{0}, HH'\right) \tag{2.34}$$

where

$$\boldsymbol{y}_t = \begin{bmatrix} r_t \\ \tilde{y}_t \end{bmatrix}; \quad d = \begin{bmatrix} 0 & 0 & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}; \quad \boldsymbol{X}_t = \begin{bmatrix} \tilde{y}_{t-1} \\ r_{t-1} \\ \tilde{q}_{t-1} \end{bmatrix}; \quad z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\alpha_2 \end{bmatrix};$$

$$\boldsymbol{A}_t = \begin{bmatrix} \bar{r}_t \\ g_t \\ \bar{r}_{t-1} \end{bmatrix}; \quad \boldsymbol{\eta}_t = \begin{bmatrix} e_t \\ \xi_t \end{bmatrix}; \quad \mathrm{V}(\boldsymbol{\eta}_t) = GG' = \begin{bmatrix} \sigma_0^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}; \quad T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix};$$

$$\boldsymbol{\nu}_t = \begin{bmatrix} \vartheta_{1t} \\ \vartheta_{2t} \\ 0 \end{bmatrix}; \quad \mathrm{V}(\boldsymbol{\nu}_t) = HH' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The state space representation of the HPMV minimization problem for the second economic relationship assuming that γ_t follows a random walk process is given by the equations (2.33) and (2.34) with the following matrices and vectors:

$$\boldsymbol{y}_t = \begin{pmatrix} i_t^{90TD} \\ i_t^{ON} \end{pmatrix}; \quad d = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \rho & (1-\rho) & (1-\rho)\alpha_1 & (1-\rho)\alpha_2 \end{bmatrix}; \quad \boldsymbol{X}_t = \begin{bmatrix} i_{t-1}^{ON} \\ E_t\pi_{t+s} \\ \pi_{t+s} - \bar{\pi}_{t+s} \\ \tilde{y}_t \end{bmatrix};$$

$$\boldsymbol{A}_t = \begin{pmatrix} \bar{r}_t \\ g_t \\ \gamma_t \end{pmatrix}; \quad z = \begin{bmatrix} 1 & 0 & 0 \\ 1-\rho & 0 & \rho - 1 \end{bmatrix}; \quad \boldsymbol{\eta}_t = \begin{bmatrix} e_t \\ \varepsilon_t \end{bmatrix}; \quad \mathrm{V}(\boldsymbol{\eta}_t) = GG' = \begin{bmatrix} \sigma_0^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix};$$

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \boldsymbol{\nu}_t = \begin{bmatrix} \vartheta_{1t} \\ \vartheta_{2t} \\ \vartheta_{3t} \end{bmatrix}; \quad \mathrm{V}(\boldsymbol{\nu}_t) = HH' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}$$

These state space representations assume, for both economic relationships, that the change in the trend component of the real interest rate, $\Delta\bar{r}_t$, is modeled as a random walk plus drift

$$\bar{r}_t = \bar{r}_{t-1} + g_{t-1} + \vartheta_{1t}$$
$$g_t = g_{t-1} + \vartheta_{2t}$$

Empirical comparisons of the two estimation methods (optimization and Kalman filter) have found different results in term of the unobserved trend component (Boone [2000]). With respect to the optimization method, the state space approach has the advantage that the estimates of the parameters and the unobserved variable are obtained simultaneously, including also standard errors for the unobserved variable. Another advantage of the state space version is that it allows different representations for the unobserved component, not only the random walk implied by the HPMV representation.

For most empirical applications $\boldsymbol{\lambda}$ is fixed by the user. The most common values for these parameters are $\lambda_1 = 1600$ for quarterly data and 14400 for monthly data and λ_2 is such that the ratio $\frac{\sigma_1^2}{\sigma_2^2}$ is between 0.1 and 0.5.

For this exercise we estimate the parameters of the economic relationships imposing some economic restrictions. A grid of values is used for λ_1 and λ_2 (or σ_0, σ_1 y σ_2). However, the estimated parameters are similar to the most common values.

2.3. **A Semi-structural Model.** This approach is based on a semi-structural model for a small open economy, following the work of Castillo et al. [2006] who estimated the NIR for Peru. We used the same system of equations, even though our version considers inflation expectations as an endogenous variable.

In this context, the NIR is defined as the interest rate that does not affect the output dynamics in the short run and assures output and inflation convergence to their long run equilibriums. This definition differs from the ones mentioned in previous subsections as it does not represent any trend component of the interest rate. In fact, as noted by Laubach and Williams [2001], the interest rate could deviate for long periods of time from the NIR as may happen during inflationary or disinflationary episodes.

The difference between the observed interest rate and the NIR is called the real interest rate gap; this variable is used (see Laubach and Williams [2001], Mésonnier and Renne [2007], Castillo et al. [2006]) as a measure for the stance of monetary policy. In this paper however, the interest rate gap is not exactly a variable that represents this definition given that we use a market interest rate instead of a policy interest rate. We do not include the latter because it may not represent the relevant interest rate for the agents in the economy, as the households and firms do not face it directly.

The model consists of an IS curve that represents aggregate demand, a Phillips curve that represents aggregate supply and some other equations that explain the dynamics of the real exchange rate. The remaining equations represent the determinants of the NIR. The openness of the economy is described by the real exchange rate gap and the terms of trade in the IS curve, the price of imports in the Phillips curve and by considering the uncovered interest rate parity as a determinant of the NIR. The system of equations can be represented in a state space form (see Appendix D), the unobserved variables are obtained by means of the Kalman filter and the hyperparameters calibrated or estimated by bayesian techniques.

The system equations are given by

IS curve

$$\tilde{y}_t = \alpha_1 \tilde{y}_{t-1} + \alpha_2 \tilde{r}_{t-1} + \alpha_3 \tilde{q}_{t-1} + \alpha_4 \tilde{T}_t + \eta_t^{\tilde{y}}$$

where $\tilde{y}_t$ is the output gap, $\tilde{r}_t$ is the real interest rate gap, $\tilde{q}_t$ is the real exchange rate gap and $\tilde{T}_t$ is the gap of the terms of trade.

Phillips curve

$$\pi_t^c = \beta_1 \pi_{t-1}^c + \beta_2 \pi_t^{imp} + \beta_3 \tilde{y}_{t-1} + (1 - \beta_1 - \beta_2) E_t \pi_{t+1} + \eta_t^{\pi}$$

where π_t^c is a measure of the core inflation, π_t^{imp} is the inflation of imported goods and services, $\tilde{y}_t$ is the output gap and $E_t \pi_{t+1}$ representes the inflation expectations for the period $t+1$ using information up to t.

Inflation expectations

$$E_t \pi_{t+1} = \bar{\pi}_{t+1} + \lambda \zeta_{t-1} + \eta_t^E$$

where $\bar{\pi}_t$ is the inflation target and $\zeta_t = (\pi_t - \bar{\pi}_t)$.

Real exchange rate gap

$$\tilde{q}_t = \rho_q \tilde{q}_{t-1} + \eta_t^{\tilde{q}}$$

Potential real exchange growth rate dynamics

$$\Delta \bar{q}_t = \varphi_0 + \varphi_1 \Delta \bar{y}_t + \varphi_2 \Delta \bar{B}_t + \varphi_3 \Delta \bar{g}_t + \varphi_4 \Delta \bar{T}_t + \eta_t^{\bar{q}}$$

where $\bar{q}_t$ is the potential real exchange rate, $\bar{y}_t$ potential output, $\bar{B}_t$ potential foreign net assets, $\bar{g}_t$ potential public expenditure and $\bar{T}_t$ is the potential of terms of trade.

Ciclycal component of Foreign net assets

$$\tilde{B}_t = \rho_b \tilde{y}_t + \rho_{ab} \tilde{B}_{t-1} + \eta_t^{\tilde{b}}$$

Potential Foreign net assets dynamics

$$\Delta \bar{B}_t = (1-\rho_{\bar{b}})b_0 + \rho_{\bar{b}}\Delta \bar{B}_{t-1} + \eta_t^{\bar{b}}$$

Real interest rate gap

$$\tilde{r}_t = \rho_{\tilde{r}}\tilde{r}_{t-1} + \eta_t^{\tilde{r}}$$

Uncovered Interest Parity

$$\bar{r}_t = \gamma_1 \bar{r}_t^* + \Delta \bar{q}_t + \tau_t$$
$$\tau_t = \gamma_\tau + \rho_\tau \tau_{t-1} + \eta_t^\tau$$

where $\bar{r}_t$ is the NIR, $\bar{r}_t^*$ is the foreign NIR and τ_t is an unobserved risk premium which is assumed to follow an AR(1) model.

Potencial GDP

$$\Delta \bar{y}_t = \phi \Delta \bar{y}_{t-1} + (1-\phi)\overline{\Delta y} + \eta_t^{\Delta y}$$

where $\overline{\Delta y}$ is the long run productivity growth.

Finally, the observed variables are decomposed as follows

(1) GDP percentage change (Δy_t)

$$\Delta y_t = \tilde{y}_t - \tilde{y}_{t-1} + \Delta \bar{y}_t$$

(2) Total inflation (π_t)

$$\pi_t = \pi_t^c + \varepsilon_t$$

(3) Percentage change of real exchange rate (Δq_t)

$$\Delta q_t = \tilde{q}_t - \tilde{q}_{t-1} + \Delta \bar{q}_t$$

(4) Percentage change in foreign net assets ($\triangle B_t$)

$$\Delta B_t = \tilde{B}_t - \tilde{B}_{t-1} + \Delta \bar{B}_t$$

(5) Nominal interest rate (i_t)

$$i_t = \tilde{r}_t + \bar{r}_t + E_t \pi_{t+1}$$

The model consists of 34 parameters, including the variances of the shocks, 17 of which were estimated by bayesian techniques using the multiple-try MCMC described in Liu et al. [2000] with the FORTRAN95 procedures developed by Bonaldi et al. [2010]. The parameters estimated are those included in the equations of the real exchange rate gap, the potential real exchange rate dynamics and the variances of the model. The others parameters were calibrated from the parameters values of the internal semi-structural model used in the central bank of Colombia (named MMT), see Gómez et al. [2002] for details of the model. They also had to match the long run equilibrium of ΔB which is calibrated at 10%, its historical average. The priors for the estimation of the variances were obtained by the calibration of the relative variability between the trend and cycle component of the variables considered in the model.

3. DATA

This section describes the data used to estimate the NIR for the methodologies presented in section 2. In all the exercises the 90-day deposit real interest rate is used to estimate the NIR. The sample period employed for the estimations is different for each methodology and depends on the availability of the required variables.

3.1. **Unobserved components models.** Two exercises are considered for the UCM. In both exercises the natural rate is defined as the trend component of the real interest rate. The first exercise is a univariate model. The second exercise is a multivariate model for the real interest rate, the logarithm of GDP and the CPI. The latter model simultaneously estimates the trend component of each variable assuming that the three series have a common level component. The series are seasonally adjusted using TRAMO-SEATS procedure (Gómez and Maravall [1996]).

The models are estimated for quarterly data using the sample period from 1982:1 to 2009:1. The inflation is measured as the annual variation of core CPI, which excludes food and administrated goods. The real interest rate is measured as the 90-day deposit nominal rate deflated by inflation expectations.

For the univariate model two measures of real interest rates are considered, RIR1 and RIR2. The first one uses caused inflation, π_{t+s}, as deflator of the nominal rate. The second one uses imperfectly rational inflation expectations (forward and backward looking), defined as $E_t\pi_{t+s} = \lambda\pi_{t+s} + (1-\lambda)\pi_{t-1}$, with $\lambda = 0.56$. The value of λ was selected according to an updated version of Gómez et al. [2002]. The measure of the real interest rate for the multivariate model is RIR1.

3.2. **Hodrick - Presscott Multivariate filter.** Two filtering exercises are performed to estimate the NIR using monthly data. Both filters are augmented by the economic relationships described in section 2.2, an IS curve and a policy rule. The IS curve exercise includes the GDP gap[3], the RIR2 real interest rate, the Real exchange rate gap defined as the deviation of the Real exchange rate index from the trend component estimated with a Hodrick-Prescott filter. The sample period is from 1980:05 to 2009:06.

On the other hand, for the policy rule, the following variables are used, 90-day deposit and the overnight nominal interest rates, total CPI inflation, GDP gap and the inflation target. Given that the inflation target is set on an annual basis, a monthly target series was estimated, assuming that the inflation target follows the same dynamic as the total CPI inflation monthly series [4]. The sample period considered for the policy rule exercise is from 1995:04 to 2009:06.

[3]The monthly GDP series is estimated applying the time series disaggregation algorithm suggested by Santos Silva and Cardoso [2001], using the quarterly GDP series and the monthly Industrial Production Index.

[4]An ARIMA model for total CPI inflation is estimated with information up to the end of each year (from 1994 to 2008) and forecasts for the following twelve months are restricted such that the target at the end of

3.3. **Semi-structural model.** For this exercise, quarterly data from 2000:01 to 2009:04 is used. The data includes the observed domestic GDP growth, total CPI inflation, real exchange rate growth, percentage change in foreign net assets and nominal 90-day deposit interest rate.

The following exogenous variables are also included in the model: percentage change of terms of trade, the terms of trade gap measured as the cyclical component obtained with the Hodrick-Prescott filter, the growth of public expenditure, inflation target, inflation of imports, the USA NIR as a proxy of foreign NIR. The latter is measured as the trend component of the 3-month certificate of deposit rate deflated by non-seasonally adjusted core inflation, where the trend component is estimated with the Hodrick-Prescott filter.

4. Results

4.1. **Univariate UCM.** The results of NIR estimation based on a univariate unobserved components model are presented in this section. This model is estimated using two definitions of the real interest rate, RIR1 and RIR2, as defined in 3.1.

The specification of the UCM for both definitions of the real interest rate is the local level plus cycle model:

$$y_t = \mu_t + \varphi_t + \epsilon_t \tag{4.1}$$

$$\mu_{t+1} = \mu_t + \eta_{t+1} \tag{4.2}$$

$$\begin{pmatrix} \varphi_{t+1} \\ \varphi^*_{t+1} \end{pmatrix} = \rho \begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} \varphi_t \\ \varphi^*_t \end{pmatrix} + \begin{pmatrix} \kappa_{t+1} \\ \kappa^*_{t+1} \end{pmatrix} \tag{4.3}$$

where $\epsilon_t \sim NID(0, \sigma^2_\epsilon)$, $\eta_t \sim NID(0, \sigma^2_\eta)$, $0 < \rho < 1$ and $t = 1, \ldots, N$.This model also includes some dummy variables for interventions in the level and the irregular components.

The estimation results and diagnostics are presented in Tables A.1, A.2, A.3 and A.4 in Appendix A. These estimations consider a high persistence of the cyclical component ($\rho = 0.9$) and a cycle period of five years, which correspond to a plausible business cycle for the Colombian economy. For both exercises, the residuals show no misspecification problems.

Figures 4.1 and 4.2 show the estimated natural interest rate using RIR1 and RIR2, respectively. Both NIR estimates reflect the trend of actual real rate; however, there are periods when they closely follow the observed series, such as the peak observed in 1998-1999, and sharp decline at the end of 1991.

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each year is achieved. Then, those restricted forecasts correspond to the estimated monthly inflation target series.

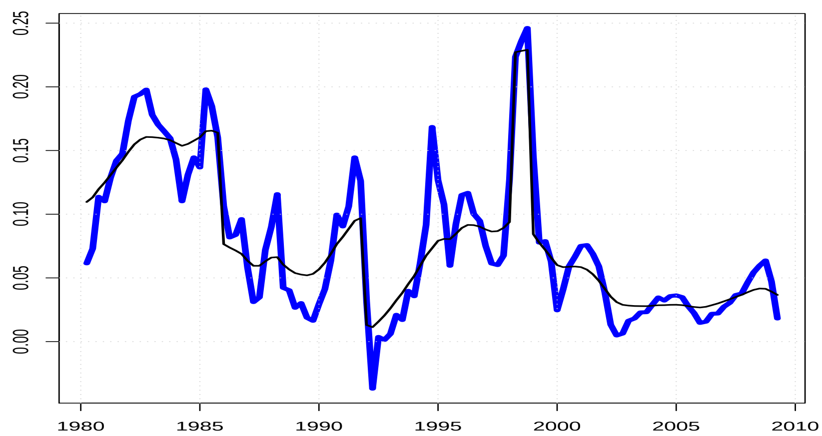

FIGURE 4.1. NIR estimation by UCM using RIR1. The real interest rate is the wide line, the NIR estimation corresponds to the thin line.

In the first half of the eighties the NIR was 13% on average, then is reduced to an average rate of 9% between 1985 and 2000 and after 2001 the NIR has been 3% in average. At the end of the sample a small increase in the NIR is observed which is reverted in the last observed quarters.

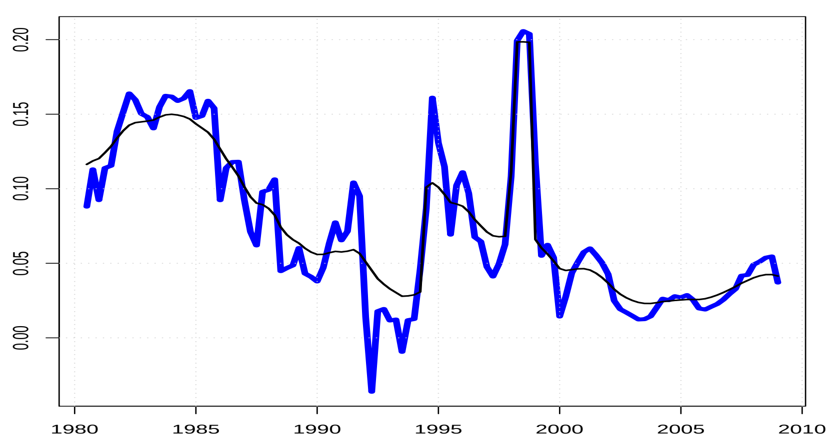

FIGURE 4.2. NIR estimation by UCM using RIR2. The real interest rate is the wide line, the NIR estimation corresponds to the thin line.

4.2. **Multivariate UCM.** The specification of multivariate UCM model that best fit the three-variable vector is the local trend plus cycle model, which is given in equations (2.23), (2.24), (2.25) and (2.16). The estimation results and diagnostics are presented in Tables A.5 and A.6 in Appendix A.

The estimated NIR and real interest rate, RIR1, are presented in Figure 4.3. The NIR dynamics is almost identical to the NIR estimation obtained in the univariate case except for the behavior around 1985, which is smoother for the univariate model. [5]

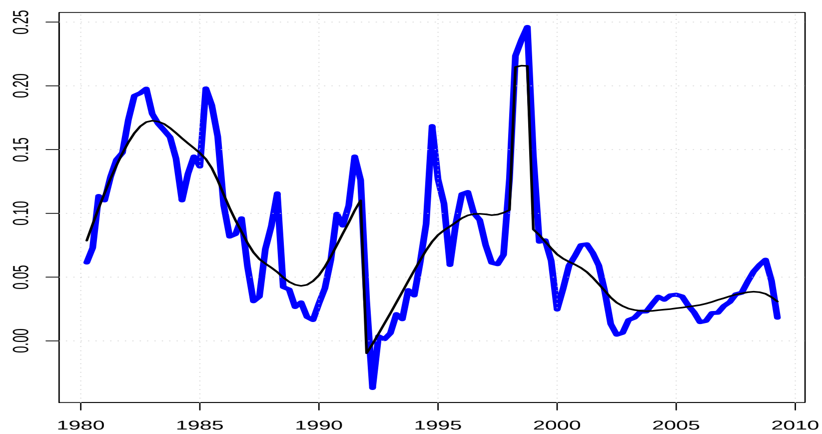

FIGURE 4.3. NIR estimation by multivariate UCM using GDP, CPI and RIR1. The real interest rate is the wide line, the NIR estimation corresponds to the thin line.

4.3. **HPMV Filter.** This section shows the estimation results based on the methodology described in section 2.2. Those estimations include two economic relationships, an IS curve and a policy rule, presented in equations (2.27) and (2.28).

4.3.1. *HPMV filter augmented by an IS curve.* For this exercise the estimated parameters obtained from the two estimation methods, optimization and Kalman filter, are very similar (Tables B.1, B.2 in Appendix B). However, the estimated NIR generated by the optimization methodology is smoother (Figure 4.4). That might be due to the difference in weight assigned to the economic relationship (parameter λ_2 in (2.26)).

With respect to the estimation of the IS curve, both the optimization and Kalman filter results show high persistence in the output gap and the exchange rate gap is not significant.

[5] An alternative exercise using RIR2 is not presented in the document since there was no specification that satisfies the residuals assumptions.

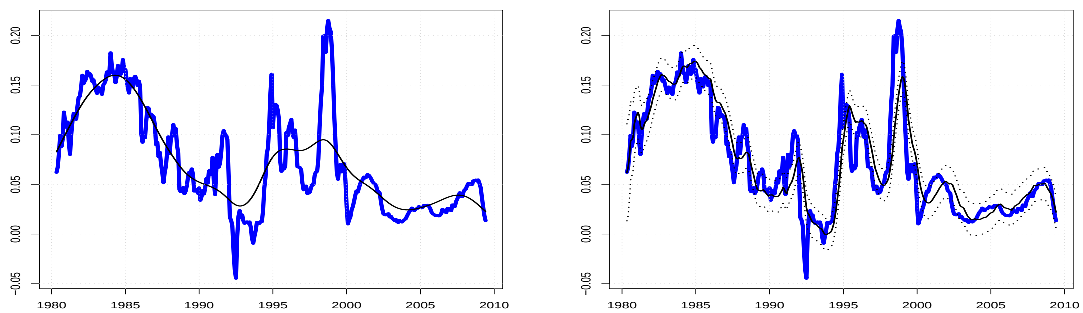

FIGURE 4.4. NIR estimation by HPMV-NLS (left panel) and HPMV-Kalman Filter (right panel) using a IS curve. The real interest rate (RIR2) is the wide line, the NIR estimation and its 90% confidence intervals correspond to the thin and dotted lines, respectively.

4.3.2. *HPMV filter augmented by a policy rule.* The estimation results for the policy rule exercise are shown in Tables B.3, B.4 in Appendix B and Figure 4.5. Although most parameter estimates are similar for both methods, optimization and Kalman Filter, there are important differences in some parameters. In the Kalman estimation the interest rate gap, the inflation gap and the overnight rate persistence are significant meanwhile in the NLS estimation (optimization method) only the overnight rate persistence is significant. As in the IS exercise, the NIR estimation by NLS is smoother than the one obtained by Kalman filter.

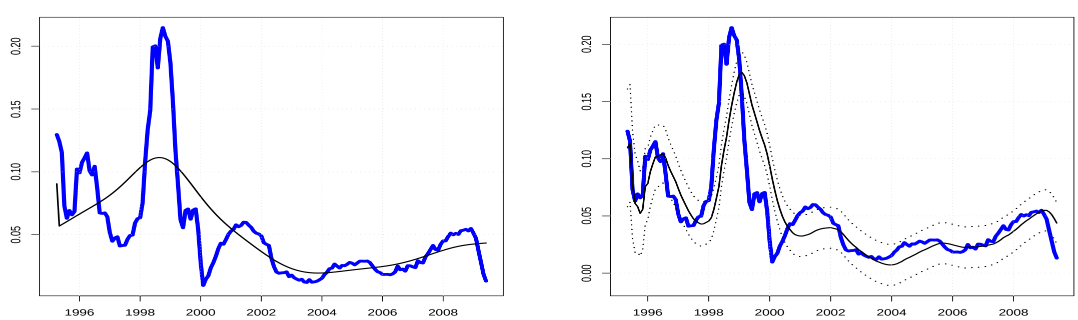

FIGURE 4.5. NIR estimation by HPMV-NLS (left panel) and HPMV-Kalman Filter (right panel) using a policy rule. The real interest rate (RIR2) is the wide line, the NIR estimation and its 90% confidence intervals correspond to the thin and dotted lines, respectively.

The dynamic of the NIR estimations with both economic relationships is similar. However, there are different trends at the end of sample, three of the four estimations show a decreasing trend, starting at different periods.

4.4. **Semi-structural model.** This section presents the results of the estimated model described in section 2.3. Here we focus our attention in the NIR estimated with the model, the details on parameters values and the estimation diagnostic is presented in Appendix D. Figure 4.6 plots the path of the real interest rate (constructed as the nominal interest rate minus the inflation expectations obtained within the model) and the natural interest rate estimated with the Kalman smoother.

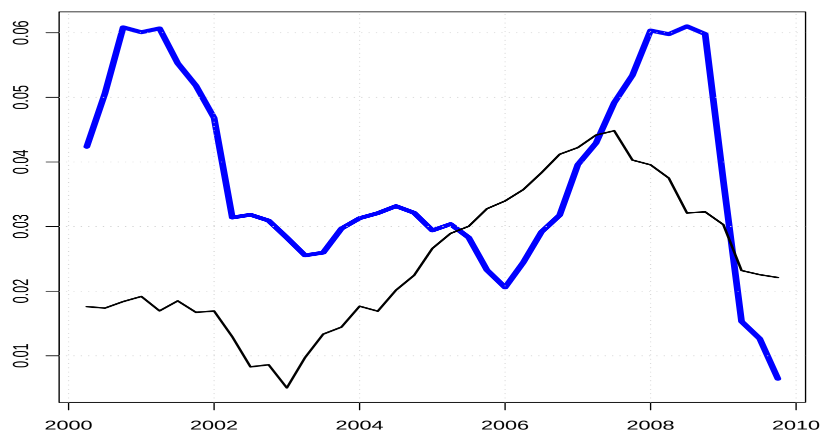

FIGURE 4.6. NIR estimation by a semi-structural model. The 90-day real interest rate is the wide line, the NIR smoothed estimation correspond to the thin line.

For the horizon considered the NIR fluctuates between 1 percent and 5 percent. It presents a positive trend since 2003Q1 to the 2007Q2, a period of sustained growth of the Colombian economy. The maximum reached is 4.4 percent in the middle of 2007. Then since 2008, as the world economy started to slowed its GDP growth, the NIR starts a decreasing trend.

The difference between the real interest rate and the NIR (interest rate gap) is a measure of the effect of the interest rate over the GDP, the IS curve of the model presents explicitly this relationship. If the real interest rate is above the NIR the interest rate have a negative effect over GDP or can be defined as a contractive interest rate. In the other case the interest rate have a positive effect over GDP and is an expansionary interest rate. Our estimation of the NIR implies that the interest rate had contractive effects on the economy during the period 2000Q1-2005Q2, and between the second quarter of 2007 and the first quarter of 2009. In the rest of the sample the interest rate had expansionary effects.

The magnitude of the contractionary or expansionary effects of the interest rate depends proportionally of the magnitude of the interest rate gap. The parameter which measures this relationship is α_2, and implies that an increase of one percentage point in the interest rate gap induces a reduction of 0.14 percentage points in the output gap. Since the interest rate gap can change due to movements in the NIR, the effect of a given level of the interest rate may change over time depending on the determinants of the NIR. Consider for example the year 2008, Figure 4.6 shows that the real interest rate stayed almost constant during that year, nevertheless as the NIR decrease the interest rate gap increase and the interest rate become more contractive at the end of the year.

Our results suggest there is significant variability of the NIR in Colombia, this is specially relevant for monetary policy because as the NIR is not observable this variability implies a large degree of uncertainty about the stance of monetary policy.

4.5. **Comparisons of NIR estimations.** Annual averages of the real interest rate and the NIR obtained by the different methodologies are presented in Table 4.1 for the period 2002Q2-2009Q2. As described in previous sections, all the real interest rate definitions use the nominal 90-day deposit rate deflated by different measures of inflation expectations. RIR1 uses observed inflation, RIR2 imperfectly rational inflation expectations and RIR.SEM inflation expectations estimated by a semi-structural model.

On the other hand, UCM1 and UCM2 denote the univariate unobserved component model estimations using RIR1 and RIR2, respectively. UCM.mv refers to NIR estimated by a multivariate UCM. The estimations obtained by the multivariate Hodrick-Prescott filter ares denoted by HPMV.$\boldsymbol{a}$.$\boldsymbol{b}$ where $\boldsymbol{a} = IS, PR$ indicates the economic relationship (IS stands for the IS curve and PR for the policy rule), $\boldsymbol{b} = NLS, KF$ refers to the estimation methodology (NLS for the non-linear optimization and KF for the Kalman Filter). Finally, SEM is related to NIR estimated by the semi-structural model.

The NIR differences are due not only to the estimation methodology but also to the Real interest rate definition.[6] Both UCM and HPMV methods estimate the long-run trend component of the real interest rate. However, the latter includes an economic relationship. On the other hand the NIR estimated by the SEM methodology is based on economic theory. This explain the smooth behavior of the NIR estimated by UCM (Figure 4.7).

The dynamics of the NIR estimations share a common pattern that can be characterized in three periods. The first part of the sample (2000-2003) shows a downward trend, followed by a period of stabilization and upward trend (2004-2008) and at the end of the sample the NIR start decreasing again. It is important to note that the SEM and the HPMV.IS.NLS estimates anticipate the fall in the last part of the sample. The NIR in the last quarter of the sample is around 3.1 in average.

[6] The real interest rates are plotted in Figure E.1 in Appendix E.

	Variable	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Real	RIR1	5.6	7.0	1.6	2.0	3.3	3.0	1.9	3.3	5.6	3.3
Interest	RIR2	4.1	5.6	2.6	1.3	2.5	2.5	2.2	3.7	5.2	2.5
Rates	RIR.SEM	5.1	5.7	3.5	2.7	3.2	2.8	2.6	4.6	6.0	2.6
	UCM1 (RIR1)	5.9	5.4	3.4	2.8	2.9	2.8	2.8	3.4	4.1	3.8
	UCM2 (RIR2)	4.6	4.4	3.1	2.4	2.4	2.6	2.8	3.5	4.2	4.1
Natural	UCM.mv (RIR1)	6.2	5.1	3.3	2.4	2.4	2.6	3.0	3.6	3.8	3.3
Interest	HPMV.IS.NLS (RIR2)	5.6	4.7	3.5	2.6	2.5	2.9	3.5	3.9	3.4	2.4
Rates	HPMV.IS.KF (RIR2)	3.4	4.6	4.2	1.9	1.7	2.7	2.7	4.2	4.9	2.9
	HPMV.PR.NLS (RIR2)	6.1	4.7	3.4	2.5	2.2	2.3	2.7	3.3	3.6	3.6
	HPMV.PR.KF (RIR2)	4.0	3.7	3.0	1.1	1.3	2.4	2.3	2.9	4.8	4.9
	SEM (RIR.SEM)	1.8	1.8	1.2	1.1	1.9	3.0	3.7	4.3	3.5	2.7

TABLE 4.1. Annual Averages of Real and Natural Interest Rates. The name in parenthesis indicates the real interest rate used in the model.

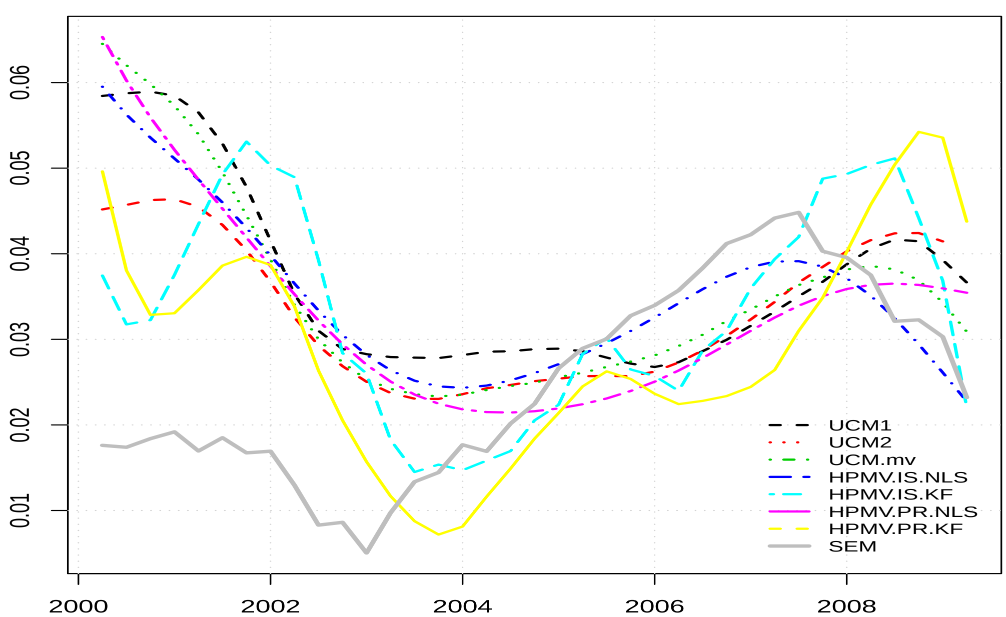

FIGURE 4.7. NIR estimations

5. Concluding remarks

This paper estimates the natural interest rate for the Colombian economy using three methodologies; unobserved component models (UCM), Hodrick-Prescott multivariate filter augmented by an economic relationship (HPMV) and a semi-structural model for a small open economy (SEM). Different definitions of inflation expectations were used in order to measure the real interest rate.

The UCM and HPMV are statistical filters and in both cases the NIR is defined as the trend component of the market real interest rate, what suggests that the NIR may be considered as a long-run real interest rate anchor for monetary policy. On the other hand, the NIR estimated by the SEM methodology is based on economic theory which considers the NIR as a medium term anchor for monetary policy.

For the common estimation sample, 2000-2009, three features are observed in the dynamics of the NIR estimates. The first part of the sample (2000-2003) shows a downward trend, followed by a period of stabilization and upward trend (2004-2008) and at the end of the sample the NIR start decreasing again. The NIR in the last quarter of the sample is around 3.1 in average.

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APPENDIX A. ESTIMATION RESULTS OF UNOBSERVED COMPONENTS MODELS

Estimated coefficients of final state vector				
Variable	Coefficient	R.M.S.E.	t-value	P-value
Level	3.6632	1.3805	2.6536	0.0091
Cycle_1	0.10049	1.0682		
Cycle_2	-0.59771	1.1487		
Lvl 1986. 1	-8.7539	1.9842	-4.4117	0.0000
Lvl 1992. 1	-8.3930	1.9850	-4.2281	0.0000
Irr 1994. 4	8.2582	2.3452	3.5213	0.0006
Lvl 1998. 2	13.342	2.0960	6.3652	0.0000
Lvl 1999. 1	-14.474	2.0955	-6.9073	0.0000
Estimated parameters of the Cycle				
The cycle variance is 1.52261				
The rho coefficient is 0.9				
The cycle period is 20 (5 years)				
The frequency is 0.314159				
The amplitude of the cycle is 0.606094				

TABLE A.1. Univariate UCM Estimation Results for RIR1

	Statistic	P-value
Skewness $[\chi^2(1)]$	0.12115	0.7278
Kurtosis $[\chi^2(1)]$	0.010216	0.9195
Normal-BS $[\chi^2(1)]$	0.13136	0.9364
Normal-DH $[\chi^2(1)]$	0.25975	0.8782
Std.Error	2.5370	
Normality	0.25975	
H(38)	0.37189	
r(1)	0.46513	
r(10)	0.11387	
DW	1.0220	
Q(10,6)	36.094	
R^2_d	0.81247	
Information criterion of Akaike		2.032894
Information criterion of Schwartz		2.268977

TABLE A.2. Diagnostic Report for the Univariate UCM using RIR1

Estimated coefficients of final state vector				
Variable	Coefficient	R.M.S.E.	t-value	P-value
Level	0.041445	0.011046	3.7521	0.0003
Cycle_1	0.0028286	0.0085470		
Cycle_2	-0.0034745	0.0091907		
Irr 1986. 1	-0.038727	0.018750	-2.0654	0.0412
Irr 1992. 1	-0.045769	0.019196	-2.3843	0.0188
Irr 1992. 2	-0.082524	0.019194	-4.2995	0.0000
Lvl 1994. 3	0.070250	0.015905	4.417	0.0000
Lvl 1998. 1	0.043588	0.020771	2.0985	0.0381
Lvl 1998. 2	0.086819	0.021153	4.1043	0.0001
Lvl 1999. 1	-0.13229	0.016821	-7.8646	0.0000
Estimated parameters of the Cycle				
The cycle variance is 9.74781e-005				
The rho coefficient is 0.9				
The cycle period is 20 (5 years)				
The frequency is 0.314159				
The amplitude of the cycle is 0.00448031				

TABLE A.3. Univariate UCM Estimation Results for RIR2

	Statistic	P-value
Skewness $[\chi^2(1)]$	0.40872	0.5226
Kurtosis $[\chi^2(1)]$	3.5748	0.0587
Normal-BS $[\chi^2(1)]$	3.9835	0.1365
Normal-DH $[\chi^2(1)]$	5.6721	0.0587
Std.Error	0.020103	
Normality	5.6721	
H(38)	0.34702	
r(1)	0.42188	
r(10)	0.062146	
DW	1.0997	
Q(10,6)	34.762	
R_d^2	0.85560	
Information criterion of Akaike		-7.605117
Information criterion of Schwartz		-7.318690

TABLE A.4. Diagnostic Report for the Univariate UCM using RIR2

Parameters*			
Variable	GDP	CPI	RIR1
Level	18.080	102.19	3.0890
	(0.000)	(0.000)	(0.1732)
Slope	0.0060225	1.2795	-0.34281
	(0.1008)	(0.000)	(0.6403)
$Cycle_1$	-0.011542	0.35084	0.0093325
$Cycle_2$	-0.0073581	-0.29772	-1.2810
Estimated parameters of the Cycle			
The amplitude of the cycle is	0.0136876	0.460139	1.28098
The rho coefficient is 0.9.			
The cycle period is 20 (5 years).			
The frequency is 0.314159			

* P-Values in parenthesis

TABLE A.5. Multivariate UCM Estimation Results using GDP, total CPI and RIR1

	GDP	CPI	RIR1
Skewness $[\chi^2(1)]$	0.85808	2.5006	5.2361
	(0.3543)	(0.1138)	(0.0221)
Kurtosis $[\chi^2(1)]$	2.5413	1.0137	8.8576
	(0.1109)	(0.3140)	(0.0029)
Normal-BS $[\chi^2(2)]$	3.3994	3.5143	14.094
	(0.1827)	(0.1725)	(0.0009)
Normal-DH $[\chi^2(2)]$	4.4787	3.3169	8.9075
	(0.1065)	(0.1904)	(0.0116)
Std.Error	0.012647	0.66210	2.5472
Normality	4.4787	3.3169	8.9075
H(38)	1.4065	37.487	0.26608
r(1)	0.057129	0.086898	0.23237
r(11)	-0.040052	-0.0023817	0.20108
DW	1.8819	1.7852	1.5078
Q(11, 6)	10.428	210.64	30.203
R_d^2	0.021200	0.35273	0.18757
Multivariate Normal DH test		15.99271	(0.0137931)

* P-Values in parenthesis

TABLE A.6. Diagnostic report for the multivariate UCM using GDP, total CPI and RIR1

APPENDIX B. ESTIMATION RESULTS OF HPMV FILTERS

Parameter	Estimate	Std. Error*	z-value	Pr(> $\|z\|$)
α_0	-0.000	0.000	-0.277	0.782
α_1	0.999	0.016	61.881	$<$ 0.001
α_2	-0.026	0.010	-2.542	0.011
α_3	-0.000	0.006	-0.031	0.976
σ_0	0.028			
λ_1	14400			
λ_2	322.591			

*VAR-COV correction matrix using Newey West - Quadratic Spectral Kernel

TABLE B.1. Optimization of the HPMV filter - IS curve

Parameter	Estimate	Std. Error	z-value	Pr(> $\|z\|$)
α_0	0.000	0.000	-0.884	0.094
α_1	0.999	0.005	207.980	0.000
α_2	-0.025	0.004	-6.799	0.000
α_3	0.000	0.002	-0.125	0.225
σ_0	0.030	0.001	25.607	0.000
λ_1	14399.993	3240.313	4.444	0.000
λ_2	346.408	42.459	8.159	0.000

TABLE B.2. Kalman Estimation of the HPMV filter - IS curve

Parameter	Estimate	Std. Error*	z-value	Pr(> $\|z\|$)
α_0	-0.017	0.032	-0.528	0.597
α_1	2.010	5.002	0.402	0.688
α_2	0.224	1.573	0.142	0.887
ρ	0.869	0.166	5.240	$<$0.001
σ_0	0.03			
λ_1	14400.0			
λ_2	1.2			

*VAR-COV correction matrix using Newey West - Quadratic Spectral Kernel

TABLE B.3. Optimization of the HPMV filter - Policy rule

Parameter	Estimate	Std. Error	z-value	$\Pr(>\lvert z \rvert)$
α_0	-0.013	0.028	-0.461	0.161
α_1	2.479	1.925	1.287	0.049
α_2	0.712	1.157	0.616	0.135
ρ	0.911	0.028	31.972	0.000
σ_0	0.031	0.001	53.036	0.000
λ_1	14399.8	5323.6	2.705	0.002
λ_2	1.036	0.027	38.812	0.000

TABLE B.4. Kalman Estimation of the HPMV filter - Policy rule

APPENDIX C. STATE SPACE REPRESENTATION OF UNOBSERVED COMPONENTS MODELS

The state space (SS) form for an unobserved component model is given by the following measurement and state equations

$$\mathbf{y}_t = Z\boldsymbol{\alpha}_t + \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim NID(\mathbf{0}, G) \tag{C.1}$$

$$\boldsymbol{\alpha}_{t+1} = T\boldsymbol{\alpha}_t + \boldsymbol{\varepsilon}_{t+1}, \quad \boldsymbol{\varepsilon}_t \sim NID(0, Q) \tag{C.2}$$

where $\boldsymbol{\alpha}_1$ is the initial state vector such that $\boldsymbol{\alpha}_1 \sim N(\boldsymbol{a}, P)$.

The measurement equation in (C.1) describes a linear relationship between the observed variables vector, $\mathbf{y}_t$, and the state vector, $\boldsymbol{\alpha}_t$. The state equation, (C.2), describes the unobserved components dynamics.

For the UCM the state vector contains the trend, seasonal and other unobserved components. The noise processes of (C.1) and (C.2), $\boldsymbol{\epsilon}_t$ and $\boldsymbol{\varepsilon}_t$, are assumed to be orthogonal to each other and serially uncorrelated Gaussian errors. They also are independent from the initial state vector $\boldsymbol{\alpha}_1$.

The SS form for the model that includes local linear trend and cycle is given by equations (C.1) and (C.2) with the following vectors and matrices

$$\boldsymbol{\alpha}_t = (\mu_t, \beta_t, \varphi_t, \varphi_t^*)', \quad \boldsymbol{\varepsilon}_t = (\eta_t, \zeta_t, \kappa_t, \kappa_t^*)'$$

$$T = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \mathbf{O} \\ \mathbf{O} & \rho \begin{bmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{bmatrix} \end{bmatrix}, \quad Z = \begin{bmatrix} 1, & 0, & 1, & 0 \end{bmatrix} \tag{C.3}$$

$$Q = \begin{bmatrix} \begin{bmatrix} \sigma_\eta^2 & 0 \\ 0 & \sigma_\zeta^2 \end{bmatrix} & \mathbf{O} \\ \mathbf{O} & \begin{bmatrix} \sigma_\kappa^2 & 0 \\ 0 & \sigma_{\kappa^*}^2 \end{bmatrix} \end{bmatrix}, \quad G = \sigma_\epsilon^2$$

where **O** represents a zero matrix of appropriate size.

The SS representation for the multivariate trend - cycle model written in SUTSE form is given by

$$\mathbf{y}_t = Z\boldsymbol{\alpha}_t + \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim NID(\mathbf{0}, G) \tag{C.4}$$

$$\boldsymbol{\alpha}_{t+1} = T\boldsymbol{\alpha}_t + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \sim NID(\mathbf{0}, Q) \tag{C.5}$$

where

$$\begin{aligned} \mathbf{y}_t &= [y_{1t}, \ y_{2t}, \ \dots, \ y_{kt}]' \\ \boldsymbol{\alpha}_{t+1} &= [\boldsymbol{\mu}_t, \ \boldsymbol{\beta}_t, \ \boldsymbol{\varphi}_t, \ \boldsymbol{\varphi}_t^*]' \\ \boldsymbol{\varepsilon}_t &= [\boldsymbol{\eta}_t, \ \boldsymbol{\zeta}_t, \ \boldsymbol{\kappa}_t, \ \boldsymbol{\kappa}_t^*]' \end{aligned} \tag{C.6}$$

$$T = \begin{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} & \mathbf{O} \\ \mathbf{O} & \rho \begin{bmatrix} \cos\lambda I & \sin\lambda I \\ -\sin\lambda I & \cos\lambda I \end{bmatrix} \end{bmatrix}, \quad Z = \begin{bmatrix} I, & 0, & I, & 0 \end{bmatrix} \tag{C.7}$$

$$Q = \begin{bmatrix} \begin{bmatrix} \Sigma_\eta^2 & 0 \\ 0 & \Sigma_\zeta^2 \end{bmatrix} & \mathbf{O} \\ \mathbf{O} & \begin{bmatrix} \Sigma_\kappa^2 & 0 \\ 0 & \Sigma_{\kappa^*}^2 \end{bmatrix} \end{bmatrix}, \quad G = \Sigma_\epsilon^2 \tag{C.8}$$

APPENDIX D. SEMI STRUCTURAL MODEL

D.1. **State - space representation.**

Measurement equation.

$$\boldsymbol{y}_t = z\boldsymbol{A}_t + \boldsymbol{\eta}_t$$

where

$$\boldsymbol{y}_t = \begin{pmatrix} \Delta y_t \\ \pi_t \\ \Delta q_t \\ \Delta B_t \\ i_t \end{pmatrix}$$

and

$$z = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{V}(\boldsymbol{\eta}_t) = GG' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma^m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Transition equation

$$w\boldsymbol{A}_{t+1} = B_t\boldsymbol{A}_t + c\boldsymbol{X}_{t+1} + H\boldsymbol{\nu}_{t+1}$$

It can be rewritten as:

$$\boldsymbol{A}_{t+1} = T_t\boldsymbol{A}_t + C\boldsymbol{X}_{t+1} + R\boldsymbol{\nu}_{t+1}$$

where $T_t = w^{-1}B_t,\ C = w^{-1}c\ ,\ R = w^{-1}H$ and

$$
\boldsymbol{A}_t = \begin{pmatrix} \tilde{y}_t \\ \tilde{r}_t \\ \tilde{q}_t \\ \pi_t^c \\ E_t(\pi_{t+1}^c) \\ \lambda \\ \tilde{B}_t \\ \bar{r}_t \\ \Delta\bar{q}_t \\ \tau_t \\ \Delta\bar{y}_t \\ \Delta\bar{B}_t \\ \tilde{y}_{t-1} \\ \tilde{q}_{t-1} \\ \tilde{B}_{t-1} \\ \Delta\bar{B}_{t-1} \\ \Delta\bar{B}_{t-2} \\ \tilde{B}_{t-2} \\ \tilde{B}_{t-3} \end{pmatrix}, \ \boldsymbol{X}_t = \begin{pmatrix} T_t \\ \pi_t^{imp} \\ \bar{\pi}_{t+1} \\ \bar{r}_t^* \\ 1 \\ \dfrac{\Delta\bar{g}_t}{\Delta Y} \end{pmatrix}, \ \boldsymbol{\nu}_t = \begin{pmatrix} \eta_t^{\tilde{y}} \\ \eta_t^{\pi} \\ \eta_t^{E(\pi)} \\ \eta_t^{\tilde{q}} \\ \eta_t^{\tilde{b}} \\ \eta_t^{\tau} \\ \eta_t^{\bar{q}} \\ \eta_t^{\Delta\bar{y}} \\ \eta_t^{\bar{b}} \\ \eta_t^{\bar{r}} \end{pmatrix},
$$

$$
w = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -(1-\beta_1-\beta_2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\rho_b & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\varphi_1 & -\varphi_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$B_t = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{\tilde{r}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta_3 & 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \zeta_{t-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \rho_{ab} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_\tau & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{\bar{b}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},$$

$$c = \begin{pmatrix}
\alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \beta_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma_1 & 0 & 0 & 0 \\
\varphi_4 & 0 & 0 & 0 & \varphi_0 & \varphi_3 & 0 \\
0 & 0 & 0 & 0 & \gamma_\tau & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (1-\phi) \\
0 & 0 & 0 & 0 & b_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

and

$$H = \begin{pmatrix}
\sigma^{\tilde{y}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^{\tilde{r}} \\
0 & 0 & 0 & \sigma^{\tilde{q}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sigma^{\pi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sigma^{E(\pi)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sigma^{\tilde{b}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma^{\bar{q}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sigma^{\tau} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^{\Delta\bar{y}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^{\bar{b}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

The variance-covariance matrix of $\boldsymbol{\nu}_t$ is an identity matrix Q of size 10×10.

D.2. **Estimation results.** Table D.1 presents a list of all the parameters and their values. For those that were estimated, the reported value is the posterior mode. The estimation were carried out using the multiple-try MCMC described in Liu et al. [2000] with the FORTRAN95 procedures developed by Bonaldi et al. [2010].

The Posterior and priors distributions of those parameters for which the likelihood function of the model was informative are reported in Table D.1

Item	Parameter	Feasible range	Value in Castillo et. al.(Ref)	Calibrated value/Posterior mode
1	β_1	$(0,1)$	0.7	0.77
2	β_2	$(0,1-\beta_1)$	0.15	0.02
3	ρ_b	$(0,1)$	0.2	0.1
4a	φ_1	$(-0.5,0.5)$	0	0.15
5a	φ_2	$(0,1)$	0.62	0.21
6	α_1	$(0,1)$	0.65	0.91
7	α_2	$(-0.5,0)$	-0.17	-0.14
8	α_3	$(0,0.5)$	0.04	0.012
9	$\rho_{\tilde{r}}$	$(0,1)$	n.a.	0.43
10a	ρ_q	$(0,1)$	0.6	0.86
11	β_3	$(0,0.5)$	0.2	0.15
12	ρ_τ	$(0,1)$	0	0.1
13	ϕ	$(0,1)$	0.95	0.95
14	$\rho_{\bar{b}}$	$(0,1)$	n.a.	0.94
15	α_4	$(0,0.5)$	0.2	0.003
16a	γ_1	$(0.4,1)$	0.65	0.66
17a	φ_4	$(-1,0)$	-0.6	-0.039
18a	φ_0	$(-5,0)$	0.45	-2.49
19a	φ_3	$(-0.1,0.1)$	0	0.007
20	γ_τ	$(0,3)$	1.25	2.34
21	b_0	$(0.05,15)$	n.a.	10.18
22a	$\sigma^{\tilde{y}}$	$(0,\infty)$	n.a.	1.88
23a	$\sigma^{\tilde{r}}$	$(0,\infty)$	n.a.	6.69
24a	$\sigma^{\tilde{q}}$	$(0,\infty)$	n.a.	26.8
25a	σ^{π}	$(0,\infty)$	n.a.	2.66
26a	$\sigma^{E(\pi)}$	$(0,\infty)*$	n.a.	0.87
27a	$\sigma^{\tilde{b}}$	$(0,\infty)$	n.a.	1.79
38a	$\sigma^{\bar{q}}$	$(0,\infty)*$	n.a.	0.019
29a	σ^{τ}	$(0,\infty)*$	n.a.	2.69
30a	$\sigma^{\Delta\bar{y}}$	$(0,\infty)*$	n.a.	0.05
31a	$\sigma^{\bar{b}}$	$(0,\infty)*$	n.a.	0.24
32	σ^{m}	$(0,\infty)$	n.a.	0.001
33	λ	$(0,1)$	n.a.	0.35
34	ρ_{ab}	$(0,1)$	n.a.	0.59

*: small variances (permanent shocks)
a: estimated parameters.

TABLE D.1. Parameter Estimations

D.2.1. *MCMC diagnostic.* Two statistics are presented in order to check convergence in the Markov chains. Figure D.2 plots the within variance of the Markov chains and a variance estimator obtained by a weighted average of the within and the between variance. They are expected to

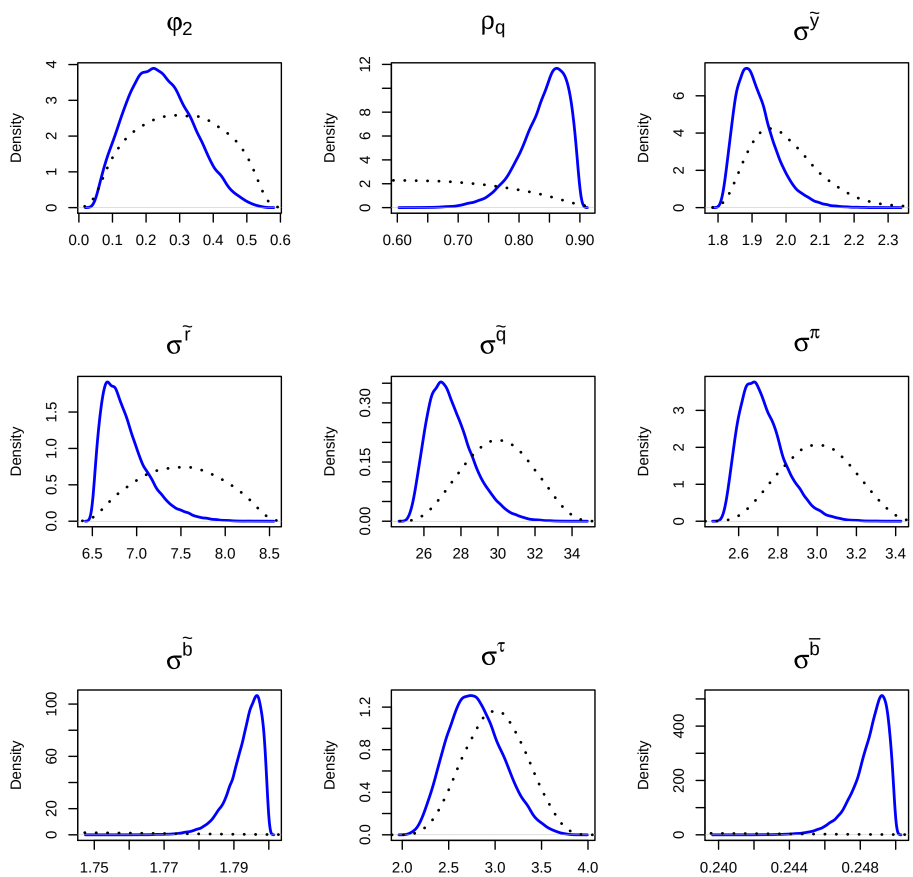

FIGURE D.1. Priors and Posterior distributions

converge both to the same level. Table D.2 presents the potential scale reductor described in Gelman and Shirley [2010] for each of the estimated parameters. Those authors argue that in practice this statistic should be less than 1.1 for each parameter (perfect mixing of the chains implies that the statistic converges to 1). Both statistics support the fact that the MCMC converged.

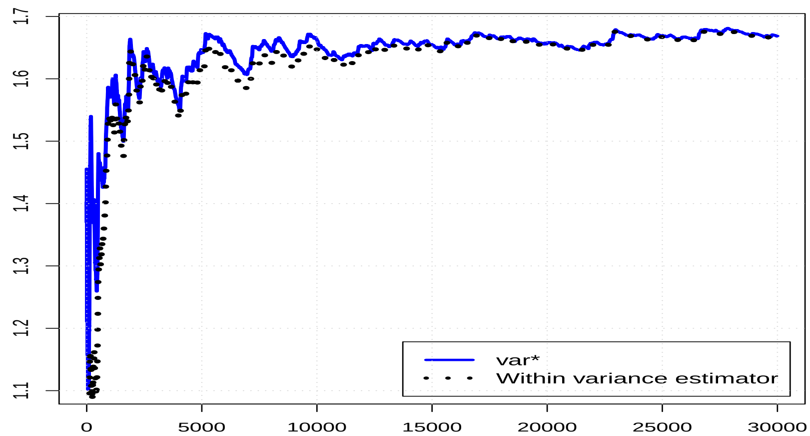

FIGURE D.2. Multivariate MCMC diagnostic

Parameter	Statistic
φ_1	1.00118
φ_2	1.00015
ρ_q	1.00036
γ_1	1.00338
φ_4	1.00097
φ_0	1.00277
φ_3	1.00169
$\sigma^{\tilde{y}}$	1.00211
$\sigma^{\tilde{r}}$	1.00210
$\sigma^{\tilde{q}}$	1.00099
σ^{π}	1.00109
$\sigma^{E(\pi)}$	1.00465
$\sigma^{\tilde{b}}$	1.00003
$\sigma^{\bar{q}}$	1.00160
$\sigma^{\Delta\bar{y}}$	1.00422
σ^{b}	1.00003

TABLE D.2. Potential scale reductor

APPENDIX E. REAL INTEREST RATES

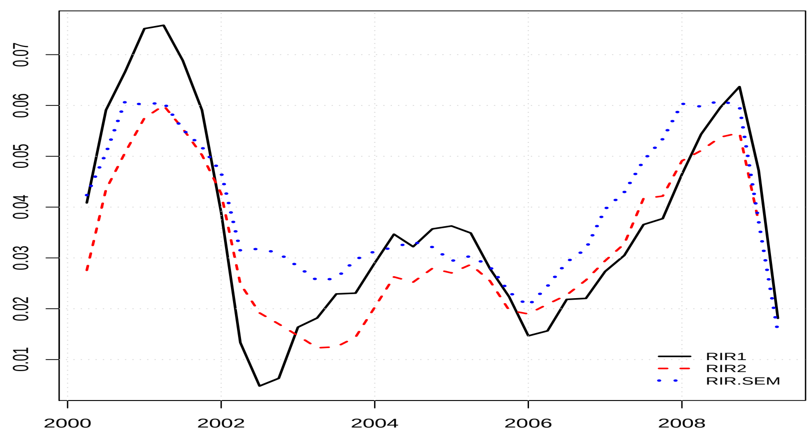

FIGURE E.1. Real interest rates