What Explains Schooling Differences Across Countries?

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Seminario Bogota

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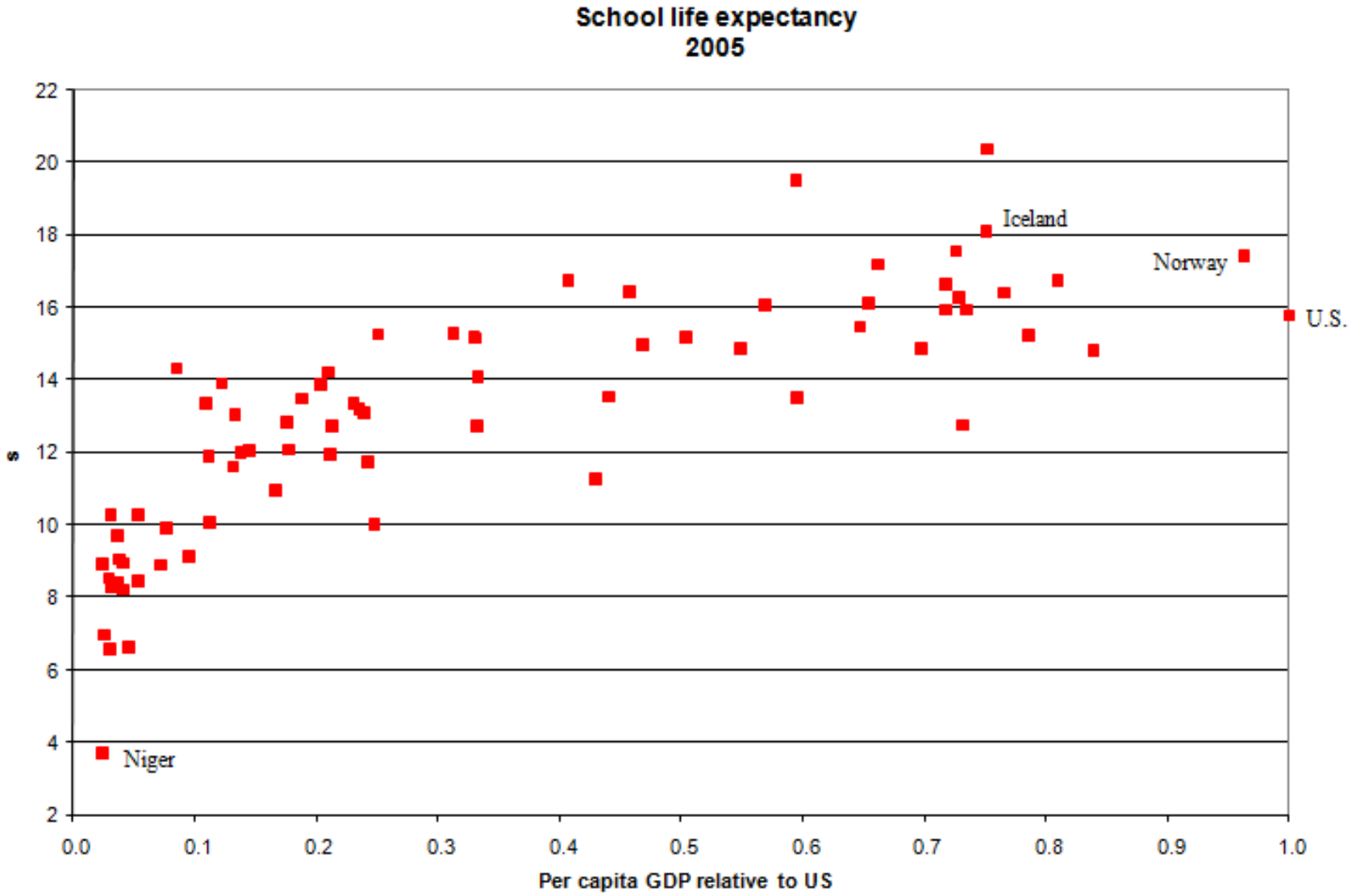
School life expectancy
2005
s
22
20
18
16
14
12
10
8
6
4
2
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Per capita GDP relative to US
Iceland
Norway
U.S.
Niger

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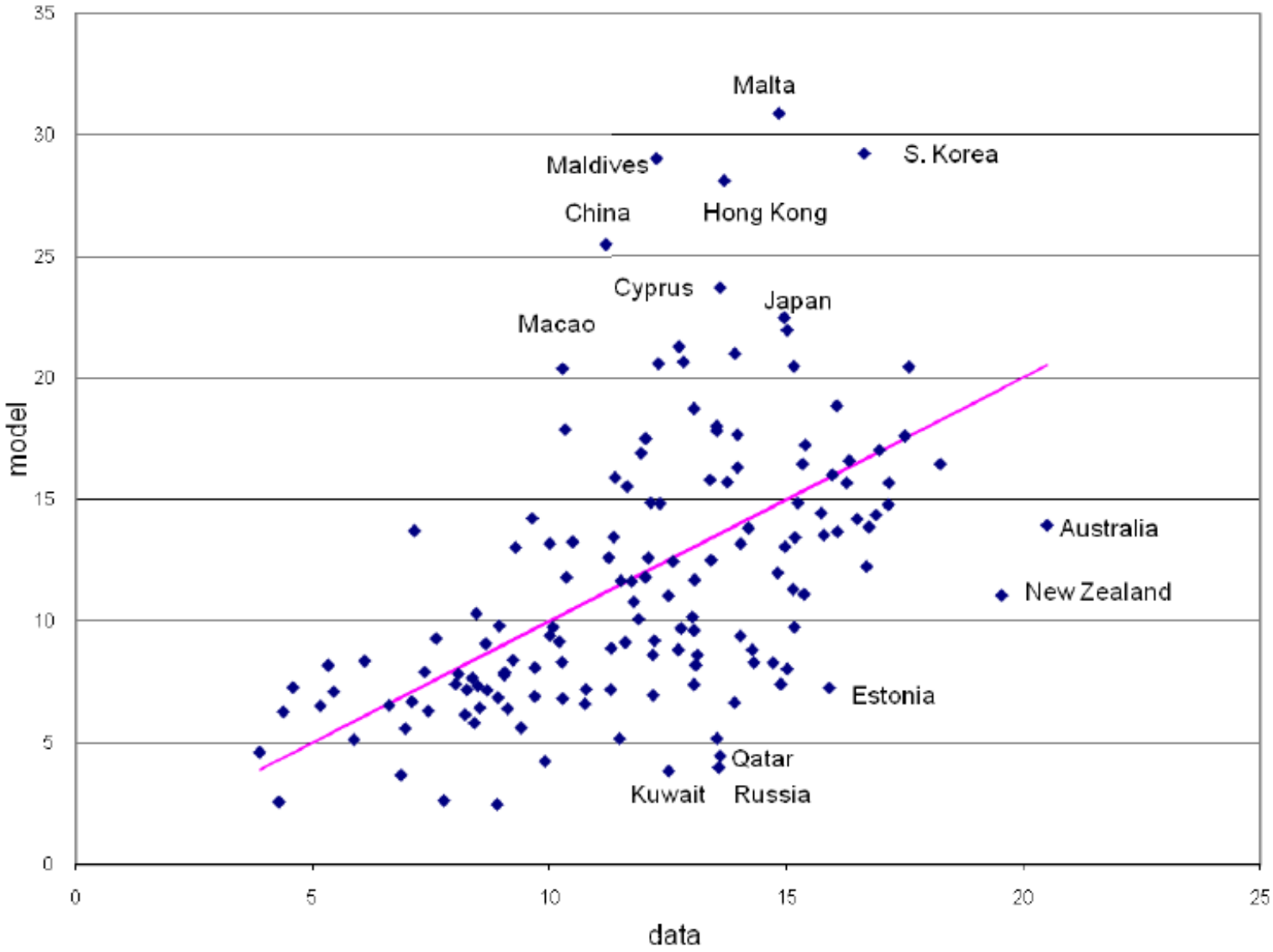
  - optimal years of schooling

$$\underbrace{(1+\mu)\, w(s)h(s)}_{\text{marginal cost}} = \underbrace{\int_s^T \left[f'(s) - g'(t-s)\right] e^{-r(t-s)} w(t)h(t)dt}_{\text{marginal benefit}}$$

Figure 1. Years of schooling - 2005
Data versus Bils and Klenow model



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  - high rate of discount: $r \geq 9.5\%$
  - downplays, by construction, the role of life expectancy

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  - cannot account for schooling dispersion

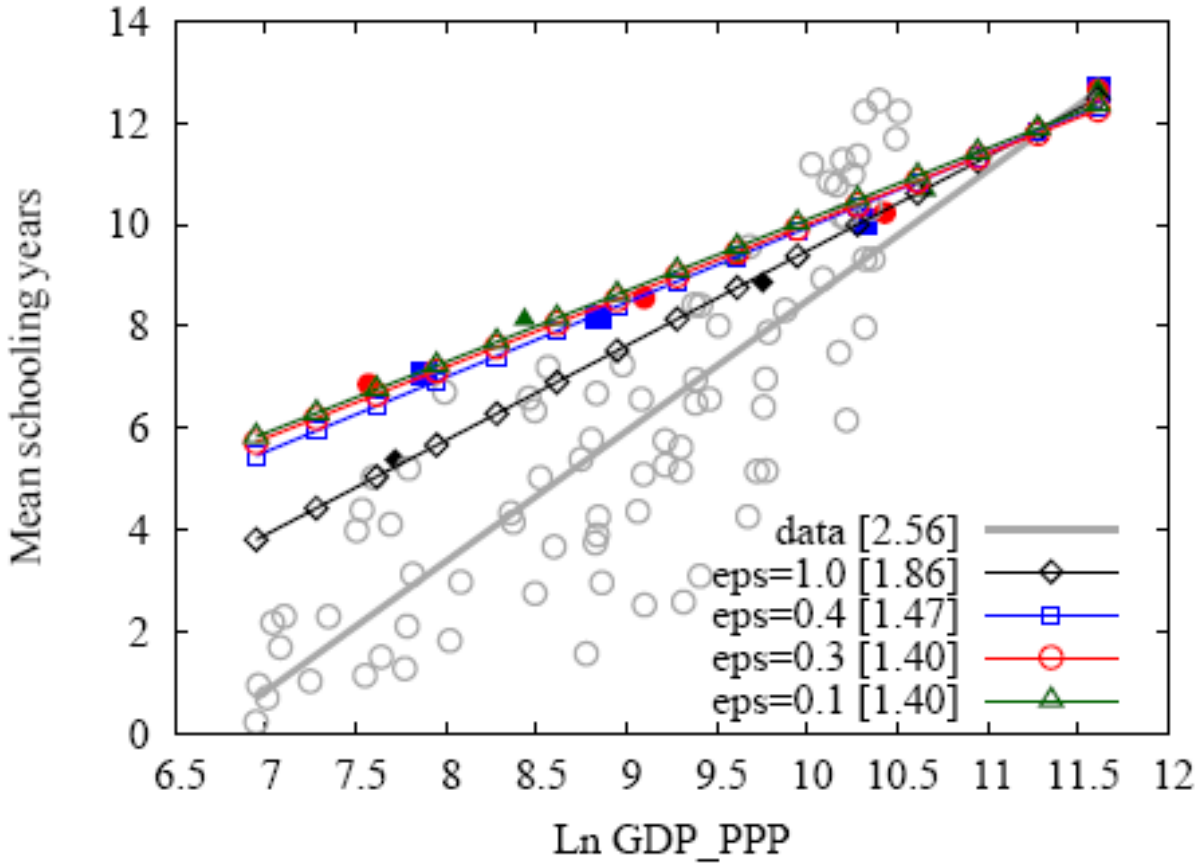

FIGURE 3

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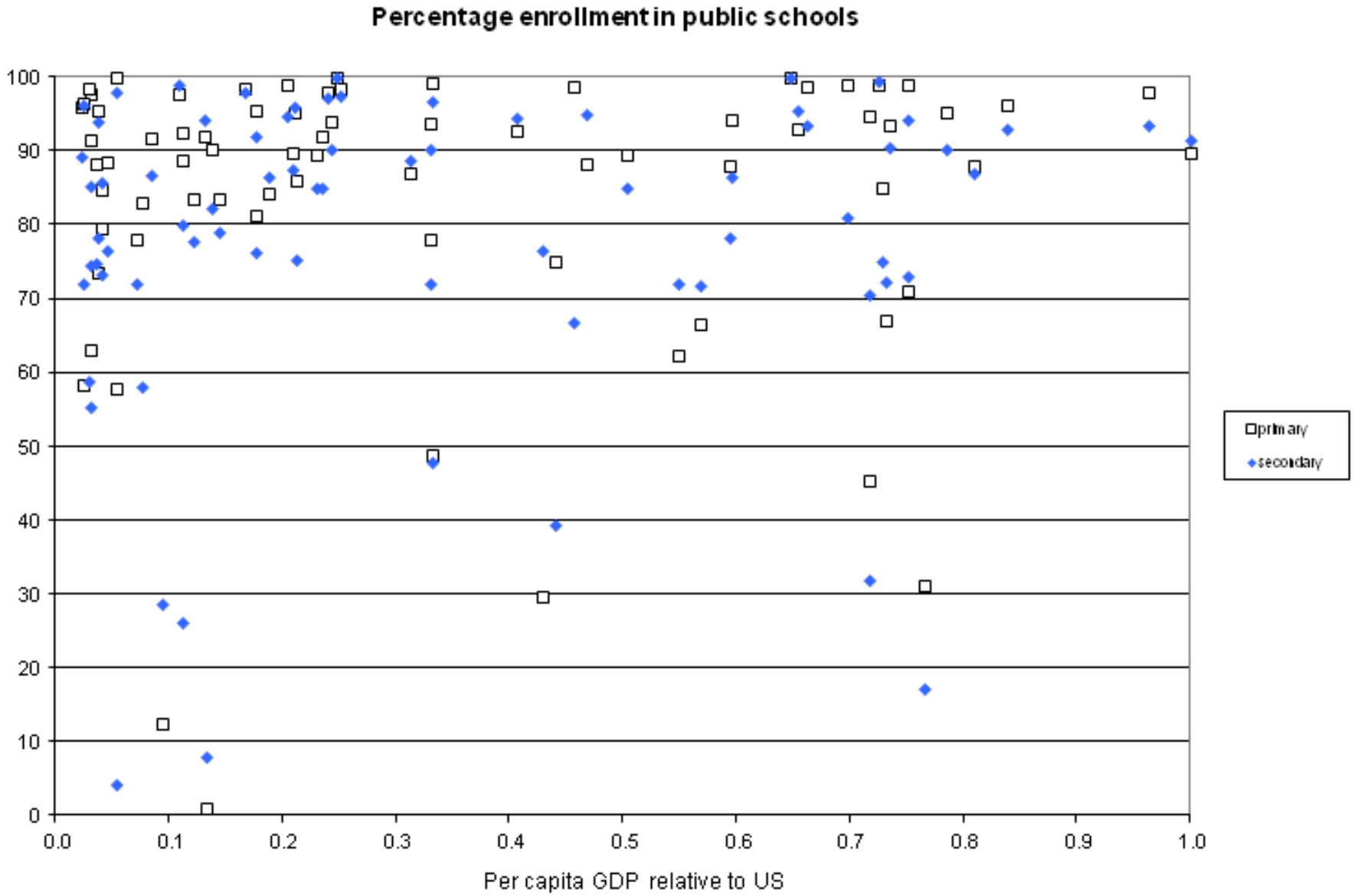
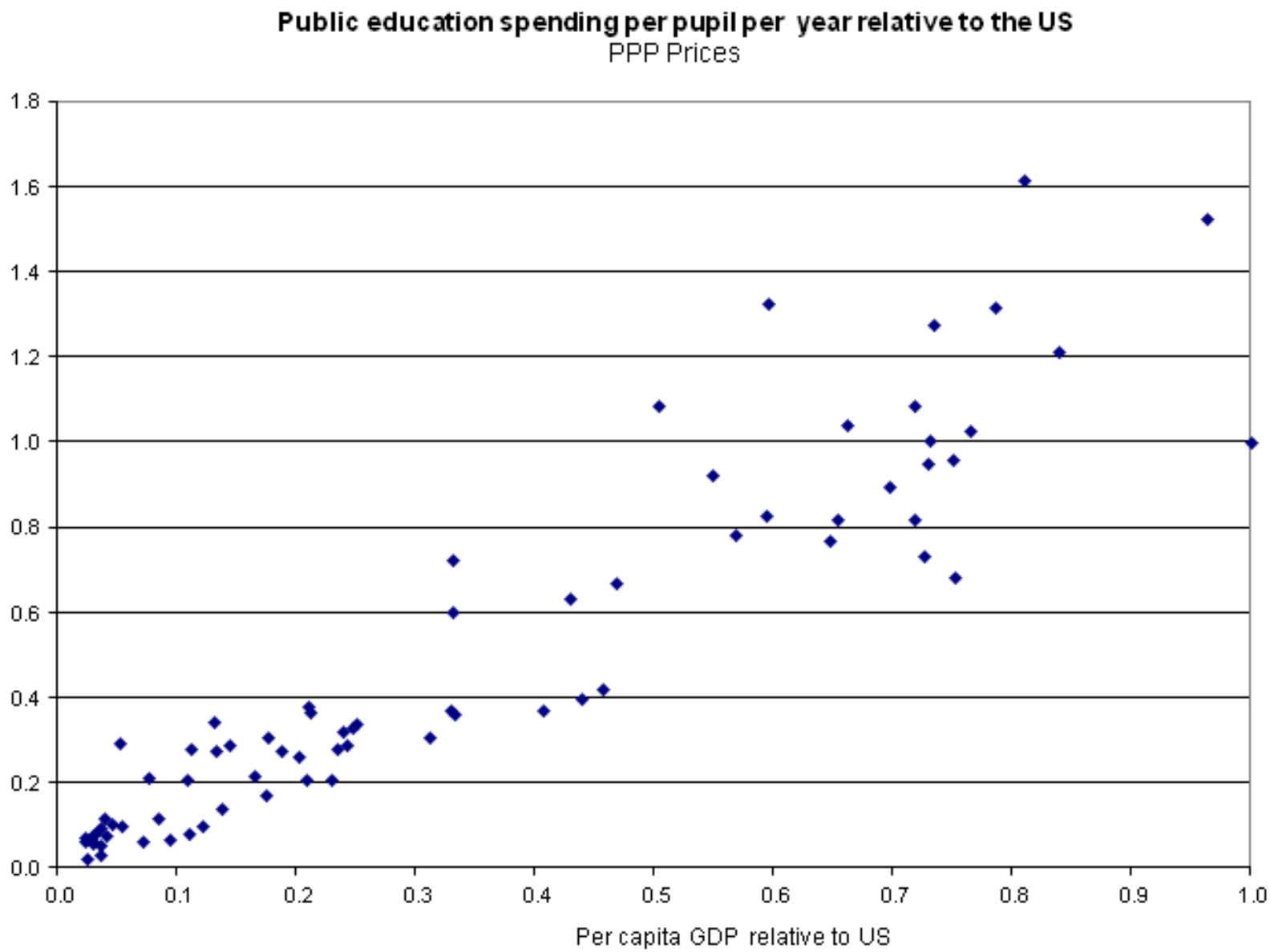
  - more spending per pupil in richer countries
  - demographic (fertility and mortality) differences across countries


Percentage enrollment in public schools
100
90
80
70
60
50
40
30
20
10
0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Per capita GDP relative to US
primary
secondary


Public education spending per pupil per year relative to the US
PPP Prices
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
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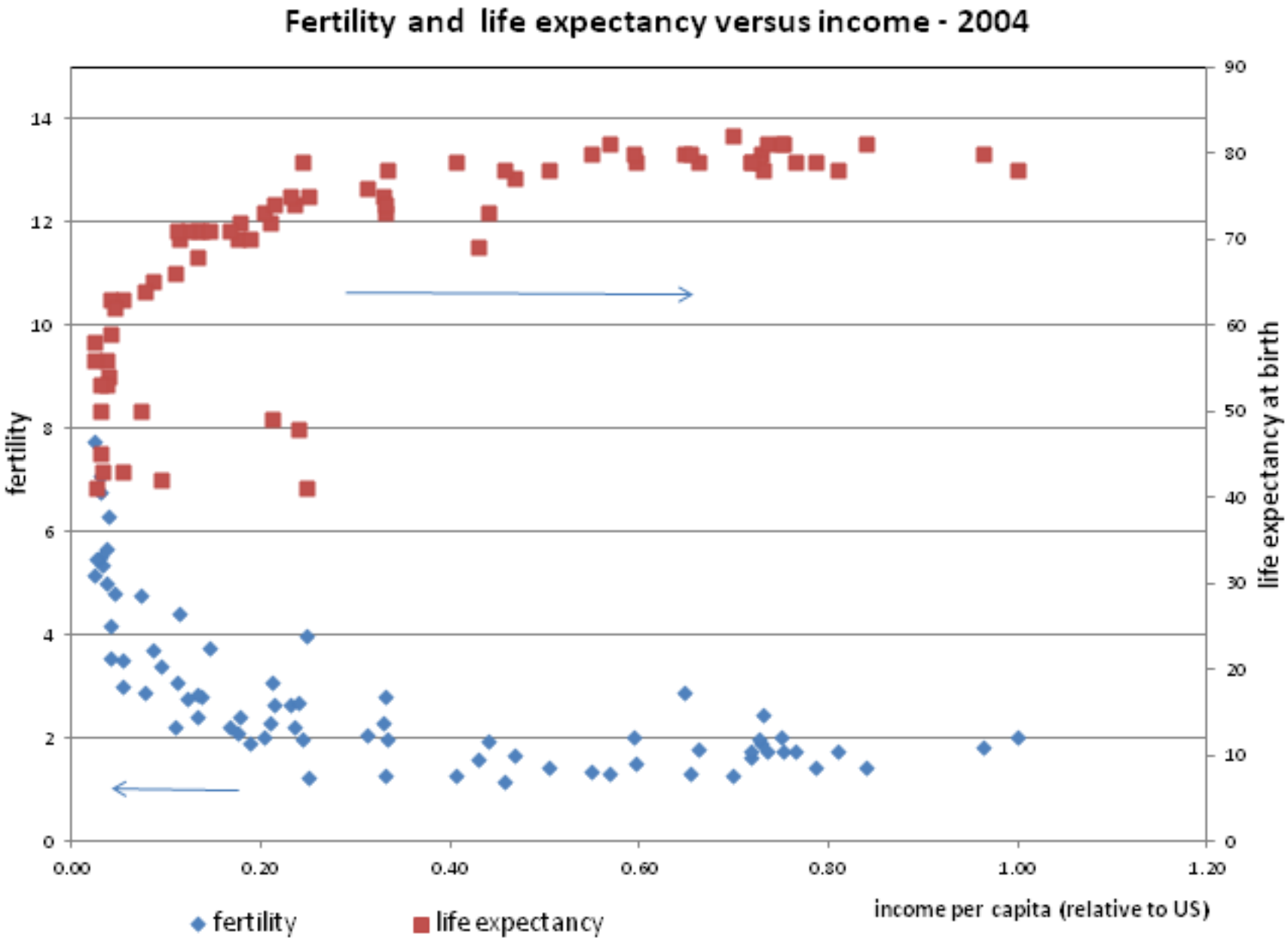
Fertility and life expectancy versus income - 2004
fertility
life expectancy at birth
income per capita (relative to US)
0
2
4
6
8
10
12
14
0
10
20
30
40
50
60
70
80
90
0.00
0.20
0.40
0.60
0.80
1.00
1.20
fertility
life expectancy

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    - benchmark model: borrowing constraints for students
    - alternative model: non-negative bequest constraint

- $h(a)$ = human capital of an individual of age a with s years of schooling

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- Students accumulate knowledge in schools according to

$$h(s) = \left(\int_0^s i(t)^\beta dt \right)^{\gamma/\beta} = \left(\int_0^s \left(\frac{e(t)}{p_E} \right)^\beta dt \right)^{\gamma/\beta}$$

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- s includes pre-school years

- Returns to schooling

$$r_s(s) = \frac{d \ln(wh(s))}{ds} = \frac{\gamma}{\beta} \underbrace{h(s)^{-\frac{\beta}{\gamma}}}_{\text{schooling}} (e(s)/p_E)^{\beta}$$

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- Example $\longrightarrow$ pure public education $e(t) = e_p$

$$h(s) = (e_p/p_E)^{\gamma} s^{\gamma/\beta}$$

$$r_s(s) = \frac{\gamma/\beta}{s}$$

- Individual's problem

$$V(b) = \max_{\substack{\{c(a),e_s(a)\} \\ s,\omega(s),b'}} \underbrace{\int_0^T e^{-\rho a} u(c(a)) \pi(a) da}_{\text{own consumption}} + \underbrace{\pi(F) e^{-\rho F} \phi(f) V(b')}_{\text{utility children}}$$

subject to

$$\underbrace{\int_0^s (c(a) + e_s(a)) q(a) da}_{\text{consumption / education}} + \underbrace{q(s)\omega(s)}_{\text{saving}} \leq \underbrace{b}_{\text{bequest}}$$

$$\underbrace{\int_s^T c(a) q(a) da}_{\text{consumption}} + \underbrace{q(F) fb'}_{\text{bequest}} \leq \underbrace{\int_s^R wh(s) e^{v(a-s)} q(a) da}_{\text{wage earnings}} + \underbrace{q(s)\omega(s)}_{\text{saving}}$$

$$h(s) = \left(\int_0^s \left(\frac{(e_p(a) + e_s(a))}{p_E} \right)^{\beta} da \right)^{\gamma/\beta}$$

$$e_s(a) \geq 0$$

$$\omega(s) \geq \underline{\omega} = 0$$

$$0 \leq s \leq F$$

$$e_p(a) = \begin{cases} e_p(a) \text{ if } \underline{s} \leq a \leq \overline{s} \\ 0 \text{ otherwise} \end{cases}$$

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  - altruism
$$\phi(f) \equiv \phi f^{\psi} \text{ with } 0 < \psi < 1$$
  - age-contingent prices
$$q(a) = e^{-ra}\pi(a)$$

- Optimality for bequest

$$\underbrace{u'\left(c\left(F\right)\right)}_{\text{marginal cost}} = \underbrace{\frac{\phi(f)}{f}u'\left(c^{child}\left(0\right)\right)}_{\text{marginal benefit}}$$

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$$G \equiv \underbrace{\frac{f}{\phi(f)}e^{-(r-\rho)F} > 1}_{\text{assumption}}$$

  - shadow price of "credit"

$$r_b = r + \ln(G)/F = \rho + \frac{(1-\psi)\ln f - \ln\phi}{F} > r$$

- Optimal education spending:

$$\underbrace{q(a)}_{\text{marginal cost}} \geq \underbrace{\frac{1}{G}\int_s^R w\frac{\partial h(s)}{\partial e^*(a)}e^{v(t-s)}q(t)dt}_{\text{marginal benefit}}$$

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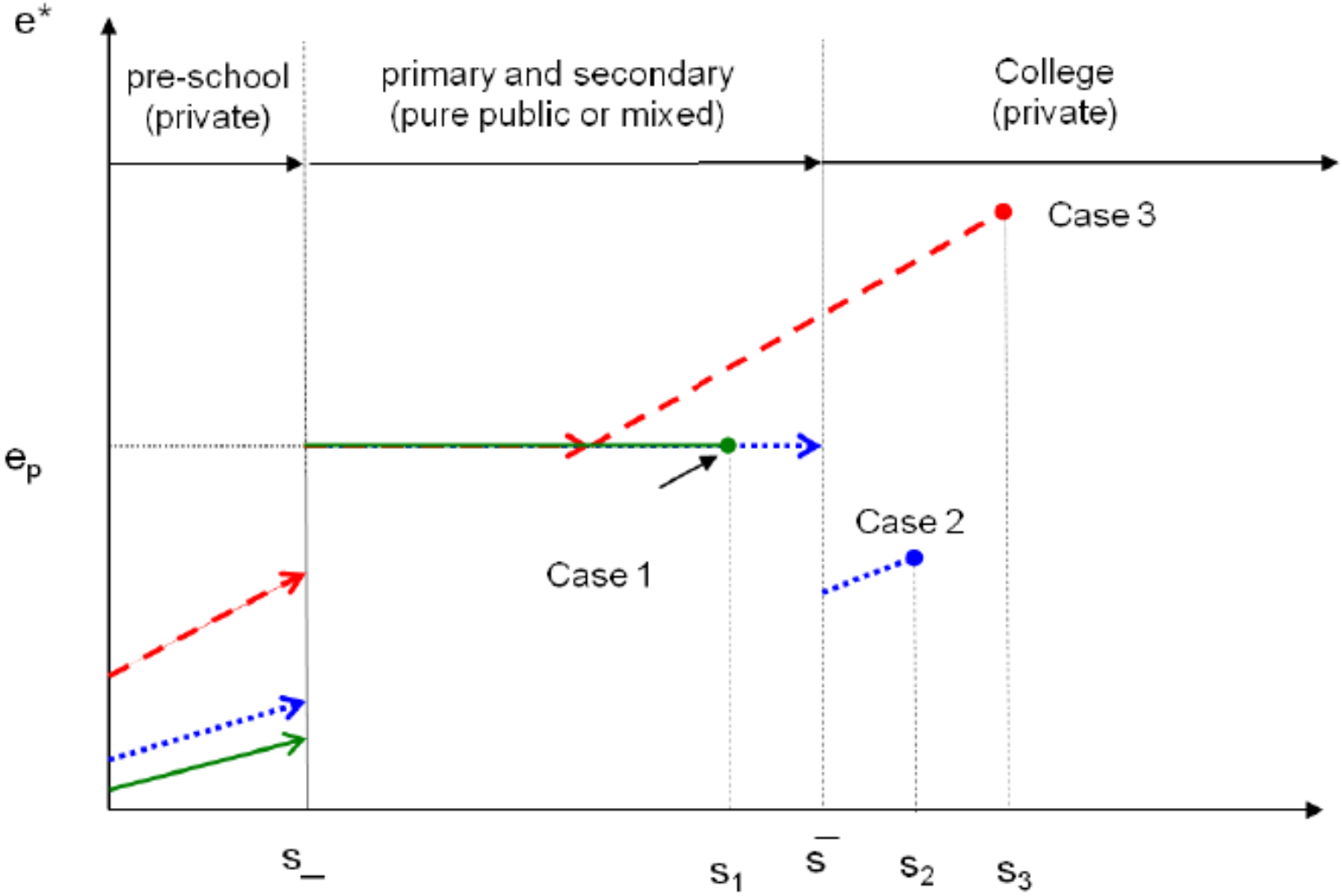
- Alternatively:

$$e^*(a) = \max\left\{\widehat{e}^*(a), e_p(a)\right\} \text{ for } a \in [0, s]$$

Figure 2. Individual expenditures in education: e*(a)



Case 1: Some public school **Case 2 :** Full public school + some private
Case 3: Full private and public school + some more private

- Optimal schooling choice:

$$\underbrace{e_s(s) + \sigma \frac{\Delta u(s)}{u'(c^S(s))}}_{\text{marginal cost}} = \underbrace{\frac{1}{G}\frac{1}{q(s)}\frac{\partial}{\partial s}\left[\int_s^R wh(s)\, e^{v(a-s)} q(a)\, da\right]}_{\text{net marginal benefit}}$$

3. Calibration

Table 1. Parameters common across countries

Parameter	Concept	Value	Source / Criteria
σ	relative risk aversion	1.5	Cooley and Prescott (1995)
ν	returns to experience	2%	Bils and Klenow (2000a)
$\underline{s}$	starting schooling age	6	UNESCO
F	parenthood age	25	Satisfies restriction $s \leq F$
R	retirement age	65	Binding level in richer countries
ϕ	level in $\phi(f) = \phi f^{\psi}$	1	Perfect altruism when $f = 1$
ψ	degree of altruism	0.4	Birchenall and Soares (2009)
r	riskless interest rate	3%	Mehra (2003)
α	capital share	0.33	Gollin (2002)

Table 2. Calibrated parameters

	Concept	Value	Target in OECD
ρ	rate of time preference	4.69%	Average schooling: 16.14 years
γ	elasticity of $h(s)$ to $e(s)$	0.3	Private spending % GDP: 0.65%
γ/β	elasticity of $h(s)$ to s	1.5	Returns to schooling: 8.28%

with years of schooling measured as:

$$SLE_a^t = \sum_{i=a}^{n} \frac{\text{enrollment}_i^t}{\text{population}_i^t} \times 100$$

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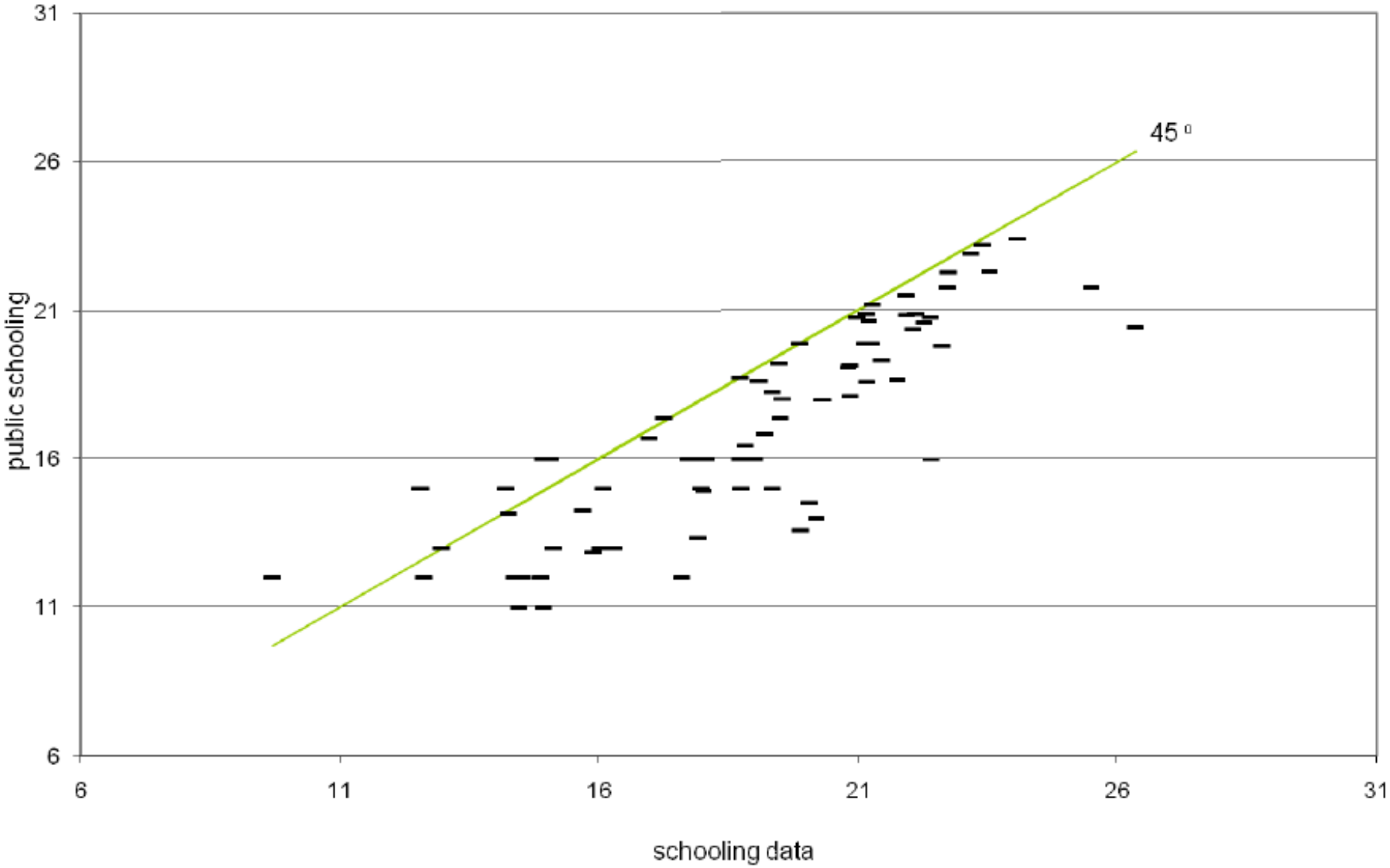
- Correction for repetition rates d:

$$h(s) = \left(\int_0^s \left(\frac{d \cdot e(t)}{p_E} \right)^{\beta} dt \right)^{\gamma/\beta}$$

Figure 3. Maximum public schooling (s upper-bar) versus school life expectancy in the data - 2005



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- Probability of survival:

$$\pi(a) = \begin{cases} e^{-p_c a} & \text{for } a \leq 5 \\ \pi(5)e^{-p_s(a-5)} & \text{for } 5 \leq a \leq 25 \\ \pi(25)\frac{e^{-p(a-5)}-\xi}{1-\xi} & \text{for } 25 < a \leq T \end{cases}$$

where $\pi(a)$ for the adult follows Boucekkine, de la Croix and Licandro (2002) and

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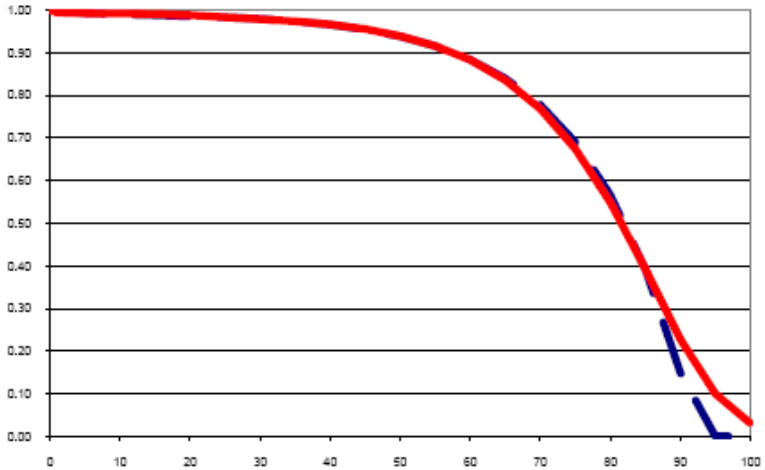
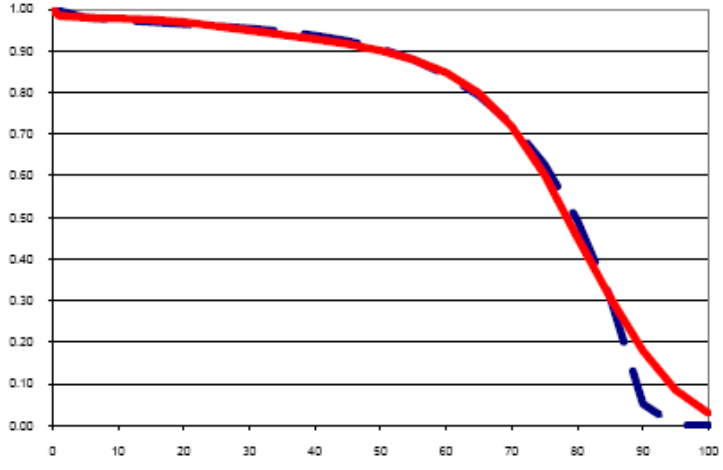
- p_E proxied by relative price of government spending from PWT

Figure 4. Survival probabilities at different ages
Precited (dashed) and Data (solid)

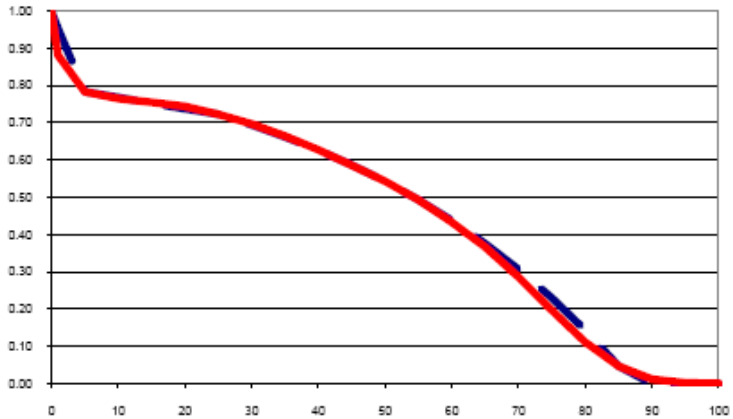
USA



Colombia



Mali



India

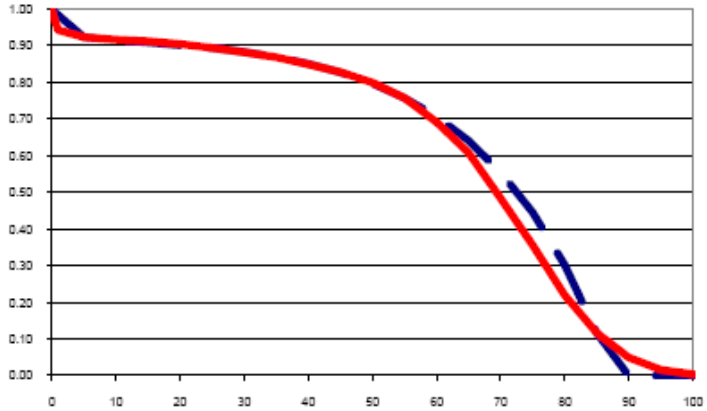


Table 3. Model's performance

	Data	Model
Means		
Years of schooling	12.96	13.60
Returns to schooling	11.2%	8.3%
Private education spending % GDP	1.2%	1.2%
Standard deviations		
Years of schooling	3.35	2.78
Returns to schooling	2.1%	1.3%
Private education spending % GDP	1.25%	0.98%
Correlation between model and data		
Years of schooling	84.7%	
Returns to schooling	86.3%	
Private education spending % GDP	35.0%	

Figure 5. School life expectancy in the model and the data

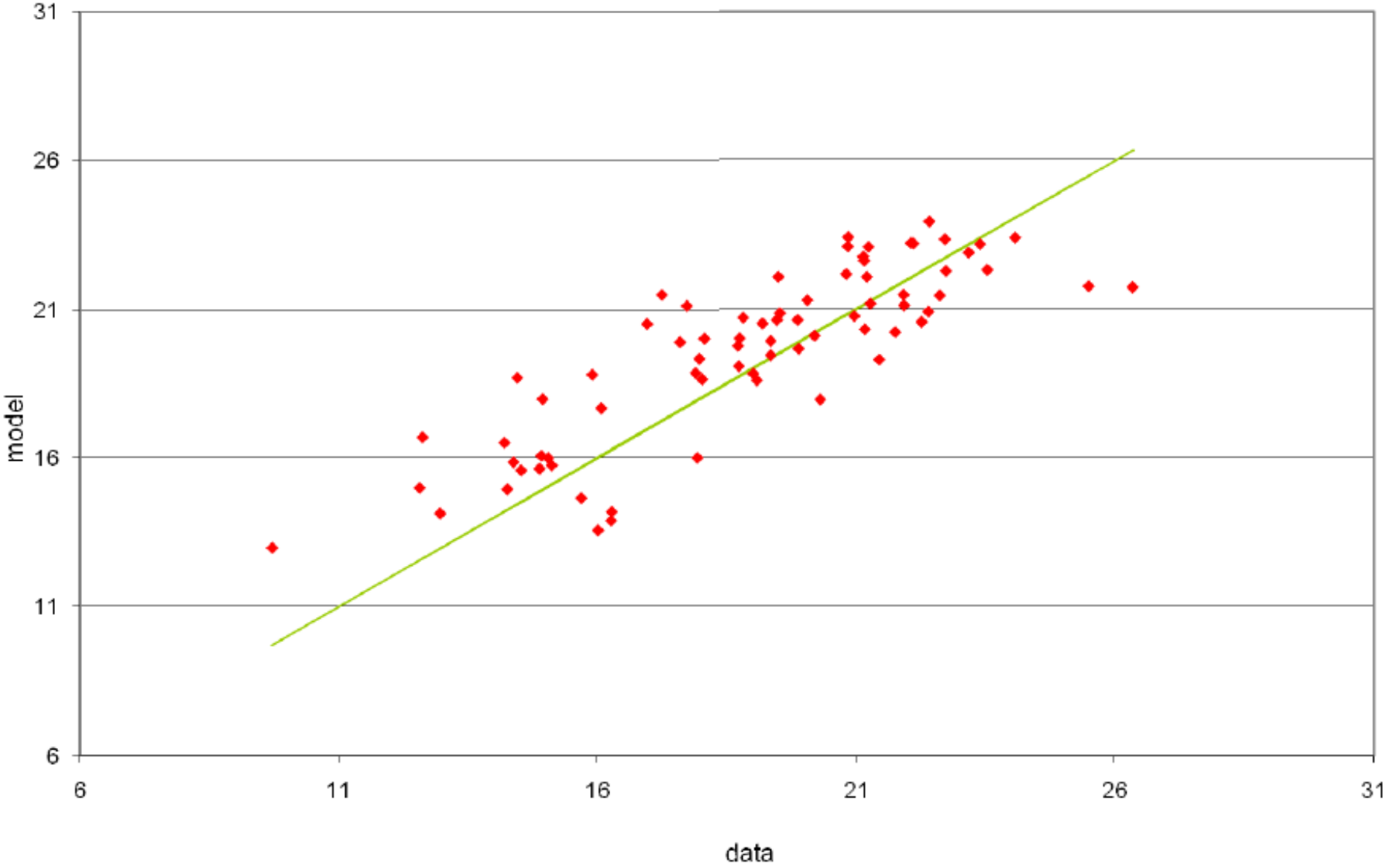


Figure 6. Returns to schooling
Model versus BK Estimates

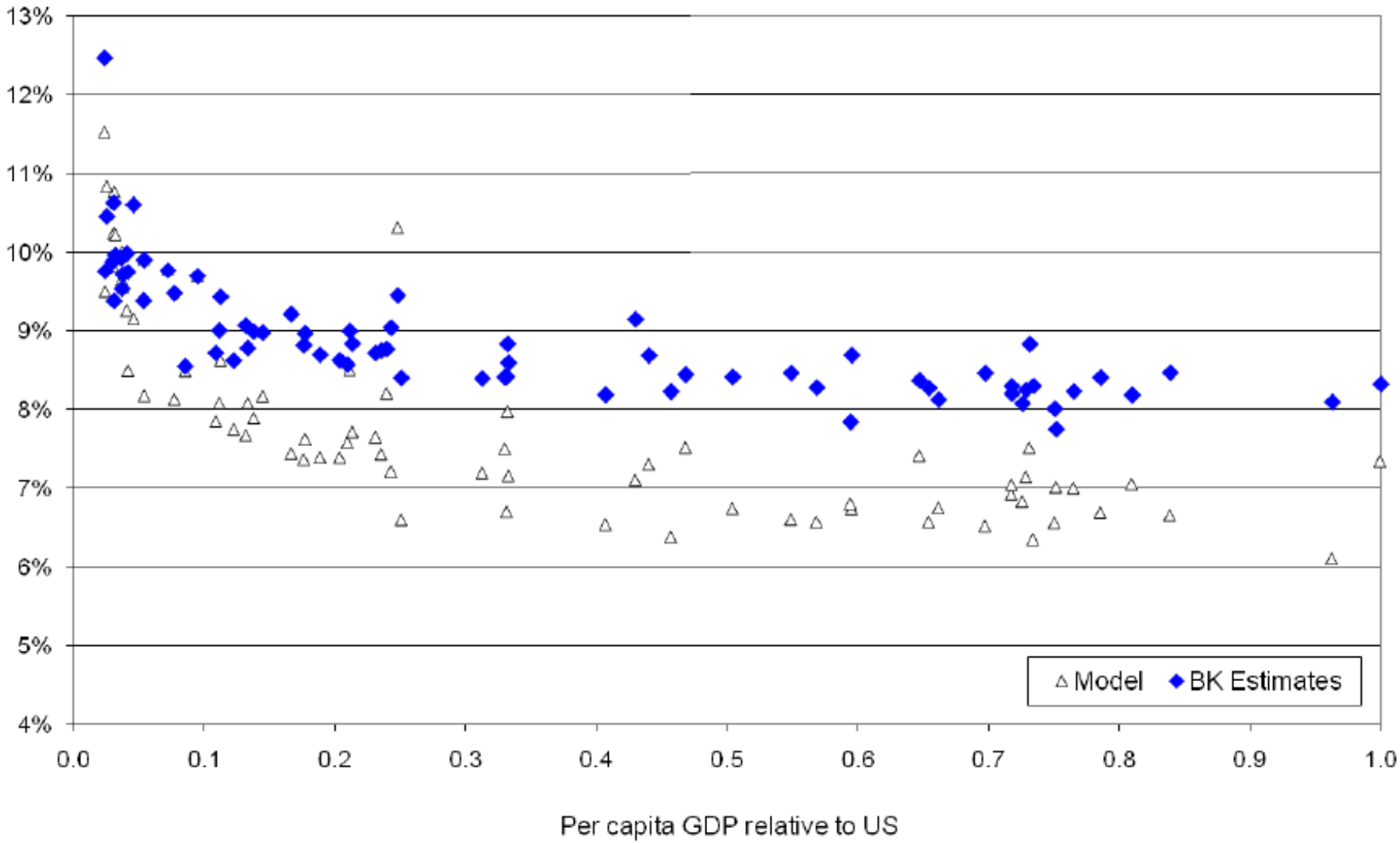


Figure 7. Private expenditures in education as a % of GDP
Model versus Data - Subset of countries

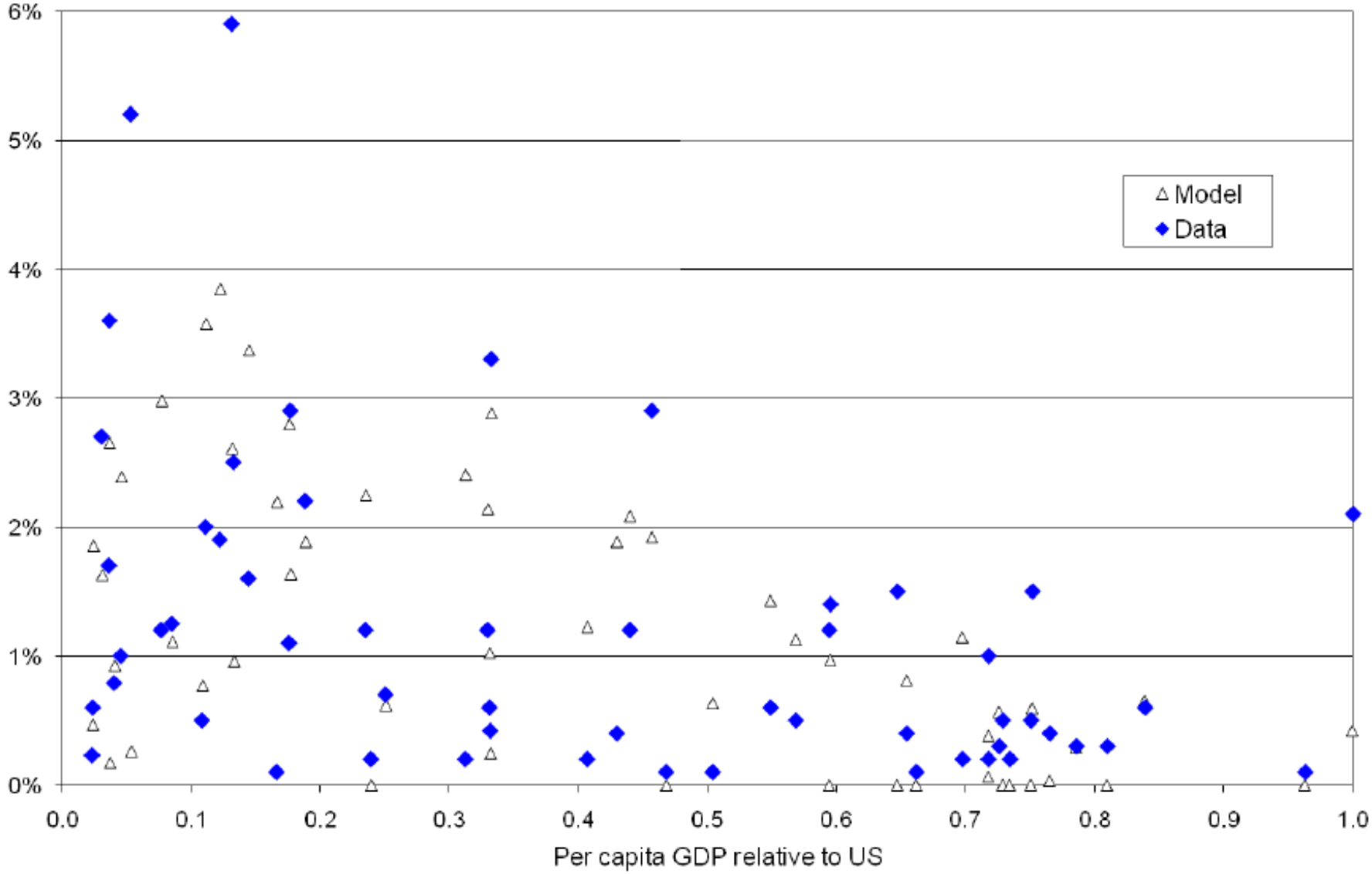


Figure 8. Quality of human capital

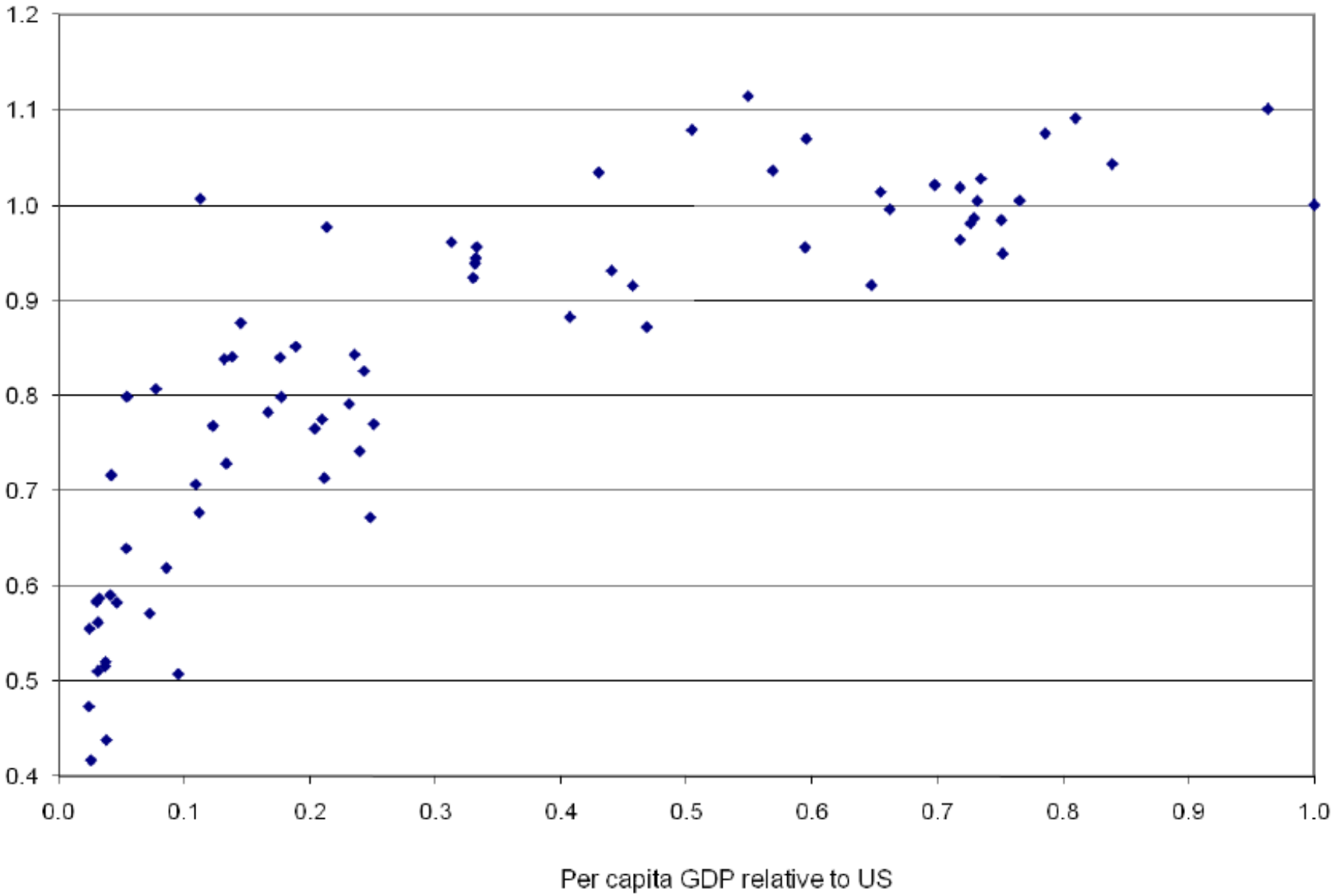


Table 4. Schooling counterfactuals (% change)

Parameter	$stdev(s)$	$mean(s)$	$var(ln(b))$	$mean(b)$
p_c	-3.7	0.4	-6.5	0.5
p_s	-3.5	0.3	-3.0	0.1
p	-22.5	2.4	-16.5	0.7
p_c, p_s, p	-30.8	3.2	-24.9	1.5
f	-56.2	3.5	-60.9	-6.4
e_p	22.7	-7.5	-17.6	2.4
$\overline{s}$	-35.3	1.8	-11.7	-1.7
p_E	2.0	-0.6	18.5	-13.0
w	-2.7	0.7	-53.7	67.2

Figure 9. Schooling: benchmark and counterfactual

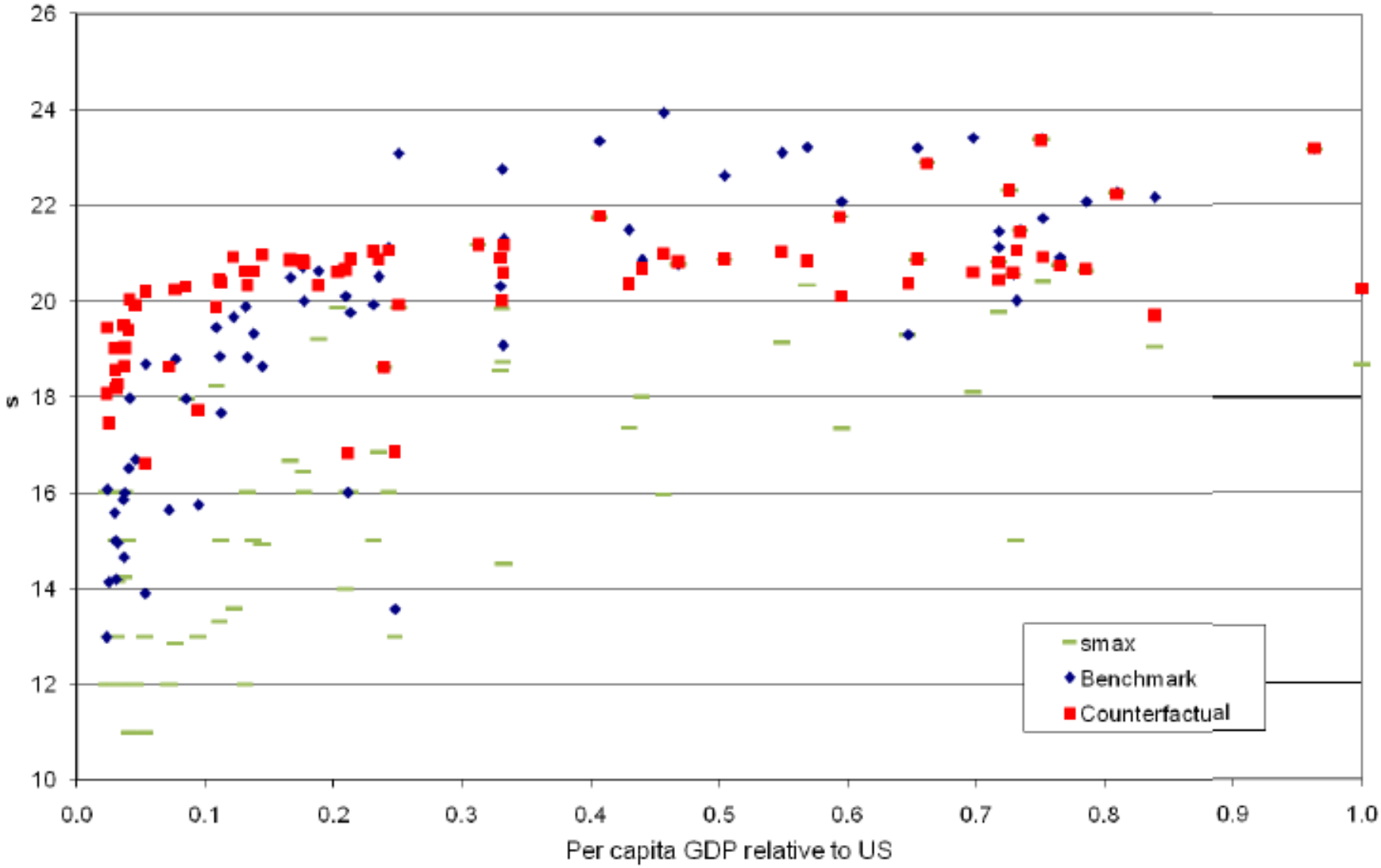
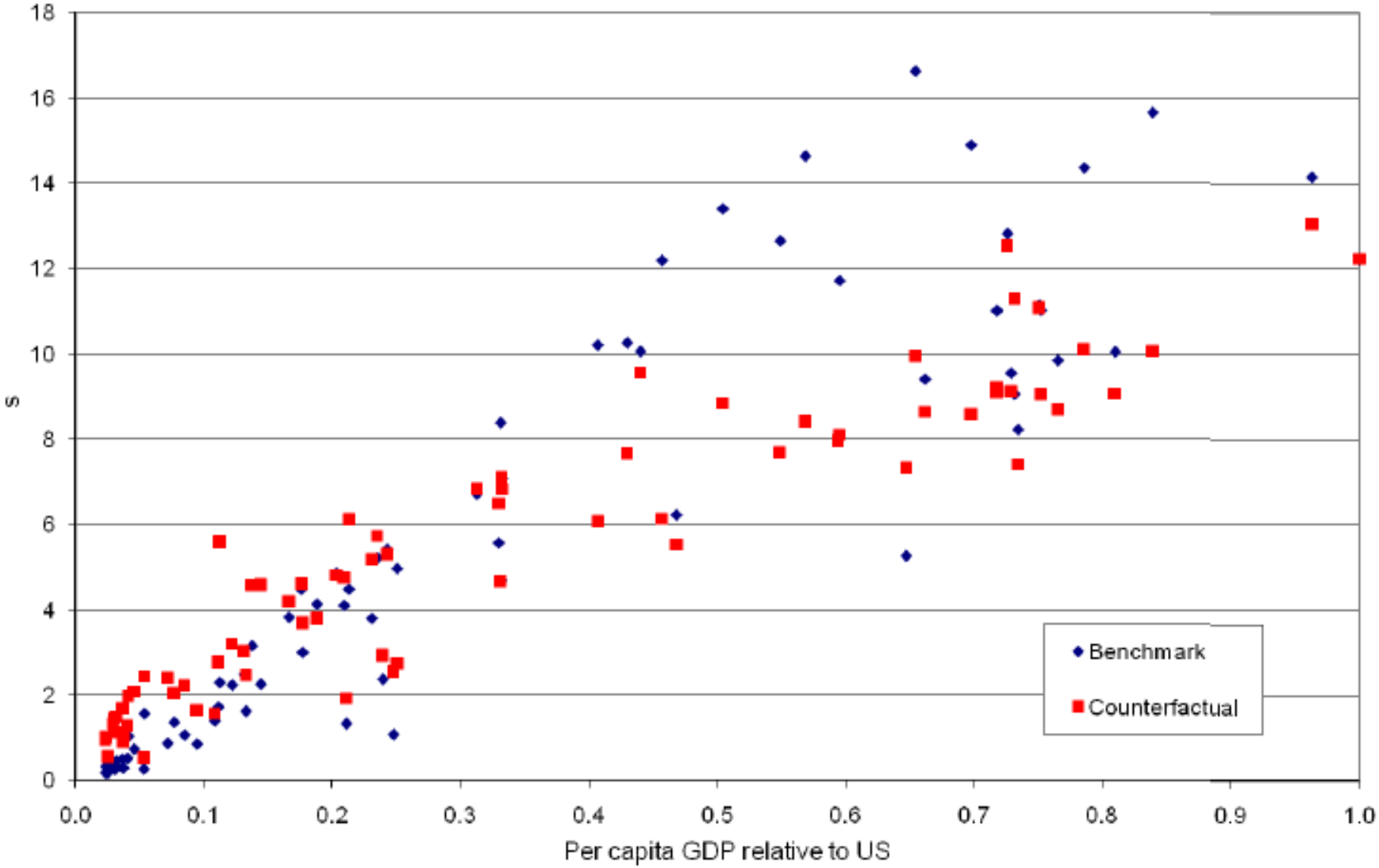


Figure 10. Parental transfers: benchmark and counterfactual



6. Robustness

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  - altruistic parameter ψ
  - frictionless version of our model
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  - low value for $\psi = 0.39$
    - model explains 94% of schooling's standard deviation

6. Robustness

- Two robustness checks:
  - altruistic parameter ψ
  - frictionless version of our model
- Altruistic parameter
  - low value for $\psi = 0.39$
    - model explains 94% of schooling's standard deviation
  - high value for $\psi = 0.58$

6. Robustness

- Two robustness checks:
  - altruistic parameter ψ
  - frictionless version of our model
- Altruistic parameter
  - low value for $\psi = 0.39$
    - model explains 94% of schooling's standard deviation
  - high value for $\psi = 0.58$
    - model explains 69% of schooling's standard deviation

6. Robustness

- Two robustness checks:
  - altruistic parameter ψ
  - frictionless version of our model
- Altruistic parameter
  - low value for $\psi = 0.39$
    - model explains 94% of schooling's standard deviation
  - high value for $\psi = 0.58$
    - model explains 69% of schooling's standard deviation
  - fertility and the duration of the public education subsidy are still key

6. Robustness

- Frictionless version of our model

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  - fertility and bequests do not play a role ($G = 1$)

6. Robustness

- Frictionless version of our model
  - fertility and bequests do not play a role ($G = 1$)
  - model can explain at most 34% of schooling dispersion

7. Alternative model

- Children live with their parents during school years

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- Children live with their parents during school years
- Parents:
  - full access to credit markets
  - make optimal consumption and schooling choices on behalf of their children
- Children become independent upon finishing school and receive a non-negative bequest
- Model has identical predictions to our benchmark model

7. Alternative model

Individual's problem:

$$V(h,s,b) = \max_{\substack{\{c(a),e_s(a)\}_{a>0}^{T} \\ s',b'}} \int_s^T e^{-\rho(a-s)} u\left(c^W(a)\right) \frac{\pi(a)}{\pi(s)} da$$

$$+\phi(f) e^{-\rho(F-s)} \left[\int_0^{s'} e^{-\rho a} u\left(c^S(a)\right) \pi(a)\, da + e^{-\rho s'} V\left(h', s', b'\right) \pi\left(s'\right)\right] \frac{\pi(F)}{\pi(s)}$$

subject to:

$$\int_s^T c^W(a)\, q(a)\, da + \int_0^{s'} f\left(c^S(a) + e_s(a)\right) q(F+a)\, da + q\left(F+s'\right) f b'$$

$$\leq \int_s^R w h(s) e^{\nu(a-s)} q(a) da + q(s) b;$$

$$e_s(a) \geq 0; \qquad b' \geq 0; \qquad 0 \leq s' \leq F$$

7. Alternative model

- Optimal bequest:

$$\frac{u'(c^S(s))}{u'(c^W(s))} = G \equiv \frac{f}{\phi(f)} e^{-(r-\rho)F} \frac{\pi(s+F)}{\pi(F)\pi(s)} > 1$$