Testable Restrictions of Nash Equilibrium in Games with Continuous Domains.

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Abstract

This paper studies the falsifiability of the hypothesis of Nash behavior, for the case of a finite number of players who choose from continuous domains, subject to constraints. The results obtained here are negative. Assuming the observation of finite data sets, and using weak, but nontrivial, requirements for rationalizability, I show that the hypothesis is falsifiable, as it imposes nontautological, nonparametric testable restrictions. An assessment of these restrictions, however, shows that they are extremely weak, and that a researcher should expect, before observing the data set, that the test based on these restrictions will be passed by observed data. Without further specific assumptions, there do not exist harsher tests, since the conditions derived here also turn out to be sufficient. Moreover, ruling out the possibility that individuals may be cooperating so as to attain Pareto-efficient outcomes is impossible, as this behavior is in itself unfalsifiable with finite data sets. Imposing aggregation, or strategic complementarity and/or substitutability, if theoretically plausible, may provide for a harsher test.

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1 Introduction:

The standard for what is to be considered scientific knowledge has been a prominent topic of debate in Epistemology. One of the most influential philosophers of the last century, Karl Popper, argued that scientists should actively try to prove their theories wrong, rather than merely trying to verify them through inductive reasoning. Hausman (1992) summarizes the methodology of Popper as three simple rules:

> "(1) Propose and consider only testable or falsifiable theories; (2) seek only to falsify scientific theories; (3) accept theories that

withstand attempts to falsify them as worthy of further discussion, never as certain..."

Accordingly, the Popperian postulate sustains that scientific discovery ought to follow four steps: (i) the internal consistency of a theory must be formally checked, to verify that it contains no logical inconsistencies; (ii) the logical principles of the theory must be distinguished from its empirical implications; (iii) the theory must be compared with alternative existing theoretical knowledge that has not been refuted by empirical evidence, in order to ascertain whether it can explain phenomena that cannot be explained by the existing knowledge; (iv) finally, the theory must be submitted to tests of its empirical implications, in order for it to be corroborated (but not verified) or refuted.[1] Interesting tests are those that are "harsh," in the sense that, *a priori*, the theory would appear likely to fail them. And if a theory fails a test, and there exists no reasonable excuse that can itself be tested, then the theory should be abandoned.[2]

This "empiricist" position, often referred to as "Falsificationism," had been previously exposed by one of the greatest mathematicians of the Nineteenth Century, Henri Poincaré, who in 1908 wrote that[3] "...when a theory has been established, we have first to look for cases in which the rule stands the best chance of being found at fault." The principle was introduced to economics by Paul Samuelson, for whom "meaningful theorems" are hypotheses "... about empirical data which could conceivably be refuted" (see Samuelson (1947), p.4).

Over the last decades, game theory has arisen as a prominent field in mathematics and economics, allowing for logically consistent and extremely elegant explanations of human behavior under interaction. The development of the theory was built largely upon the concept of Nash equilibrium, which came to be applied in many different problems, and extended and refined in many different ways. Nonetheless, much fewer attempts to derive harsh empirical tests of these developments have appeared in the literature. In terms of the Popperian rules, this would cast doubt on knowledge derived from applications of game theoretical concepts, which should not be treated as scientific, for only testable ideas deserve such treatment. In terms of the steps of scientific discovery presented in the introduction, game theory still needs to strengthen its position by completing step (ii) and then allowing for applications of step (iv).

[1] Godel´s undecidability principle shows that within the logical scope of any axiomatic theory there exist propositions whose truth or falsehood cannot be established based only on the axioms of the theory (the principles of point (ii)). One should not conclude from this that every theory is unfalsifiable: falsifiability requires that logical propositions, which can actually be proven from the logical principles, should be contrastable with reality, so as to ascertain the refutability of the logical principles of the theory. Those propositions whose logical validity cannot be ascertained from the axioms of the theory are out of the scope of the falsifiactionist method.

[2] The Popperian view has found strong criticism. For example, Hausman (1992) criticizes the position that one should *only* try to falsify theories, treating them as nothing more than "...conjectures ... worth of criticism," which Hausman finds "outrageous" (see chapter 6. Quotations are from page 81.) For a defense of (some of) the Popperian postulate, see Hutchison (1994).

[3] See Zalah (2001).

Even if one considers the views of Popper to be too extreme,[4] it seems desirable to obtain testable implications from game theoretical concepts that can prove inadequate their applications to specific problems. This paper studies the existence testable implications of the application of Nash equilibrium to an abstract situation in which finitely many agents individually choose from continuous domains, subject to constraints, and their satisfaction depends on the collective choice. I assume all the principles behind the concept of Nash equilibrium, namely that each agent acts as if he were choosing a most satisfying alternative, according to some preferences, while taking as given what the others are, in effect, doing. Preferences, of course, are not subject to tests, since they cannot be observed. But if one assumes that choices are observed, the following question can be asked: when can one say that the principles of Nash behavior cannot explain the observed choices? I propose an answer to this question in the form of testable restrictions and study how harsh a test these restrictions provide for.

The paper is organized as follows. In section 2, I survey the existing results that are related to this paper, and point out the differences between the problems addressed here and there. Section 3 explains the specific problem that I deal with, derives the testable implications of my application of the Nash equilibrium concept, studies how harsh these restrictions are and whether one could find harsher tests. Section 4 then compares the empirical implications of the application of the Nash equilibrium concept with the ones that arise from a prominent alternative hypothesis, namely that individuals cooperate to always choose Pareto-efficient outcomes. Section 5 studies whether imposing additional conditions, whose validity would have to be evaluated in specific applications, can result in harsher tests of our theory. Section 6 offers some concluding remarks.

2 Review of the literature:

There exist few results regarding the existence of testable restrictions of applications of game theoretical concepts. In unpublished work, Zhou (1999) studies the following problem: suppose that there are two individuals, each of whom chooses form some (fixed) interval from the real line, and suppose that one is given a finite subset of the Cartesian product of the two intervals. Zhou determines conditions on this set, which are equivalent to the existence of individual preferences such that the "observed set" is the Nash set of the game played by the two individuals if they have those preferences.[5] Given that there do exist sets that violate Zhou's condition, one concludes that, upon observation of the

[4] Modern defenders of Falsificationism maintain more moderate positions. See Hutchison (1994).

[5] Zhou assumes that preferences are representable by utility functions that are continuous and quasiconcave in own action. The condition that he finds, which he calls "no improper crossing," implies that the best response functions of individuals do not intersect at points other than the elements of the set.

finite set, the hypothesis that these observations, and only these observations, may come from Nash behavior is refutable.

A similar problem, for the case of finite domains, is considered by Sprumont (2000), who studies a game played by a finite set of players, each of whom can choose from a finite set. Suppose that one constructs a collective budget by choosing for each individual a nonempty subset of his choice set and then taking their Cartesian product, and that one "observes" a nonempty subset of it, as being chosen as the collective outcome. If we did this for all possible collective budgets we could construct an outcome correspondence mapping the collection of all collective choices to their chosen subsets. This correspondence would depend only on observables. The question that Sprumont answers is: what conditions does this correspondence have to satisfy if for each individual there exists a preference relation over the collective choice set such that the correspondence maps each collective budget to its Nash set under those preferences. Sprumont finds conditions that are necessary and sufficient for the rationalizability of the outcome correspondence in the mentioned sense.[6] Given that there are correspondences that violate these conditions, Sprumont concludes that, upon observation of the outcome correspondence, the application of Nash behavior to this kind of problems is falsifiable.

The analysis in Sprumont focuses on normal form games. The almost perfect analogous for extensive-form games with perfect information is Ray and Zhou (2000). Suppose that we have a finite, extensive-form game, and that we can observe an outcome function mapping the set of reduced games[7] of the original game into the set of its terminal nodes. Zhou and Ray study conditions under which, for each individual, there exists an order over the terminal nodes such that the image of each reduced game under the outcome function is its subgame-perfect equilibrium if individual preferences were precisely those orders.[8] Ray and Zhou also show that their conditions are independent, and that examples that violate one of them are not rationalizable in the mentioned sense, even if they satisfy the other two. The conclusion is again that, upon observation of

[6] The first condition, "persistence under expansion," is satisfied if whenever an outcome is chosen in two different members of the collection of collective budgets, it is also chosen in the smaller member of this collection that contains their union. The second condition, "persistence under contraction," says, subject to some qualification, that when going from a larger to a smaller member of the collection, the outcomes chosen in the larger member that "survive" the contraction should also be chosen in the smaller member.

[7] A reduced game is defined by a subset of the original three that contains some terminal nodes, and all the nonterminal nodes that belong to paths leading to them.

[8] There are three necessary and sufficient conditions. The first one, which Ray and Zhou call "acyclicity," says that for each player the (incomplete) binary relation of revealed preferences derived from the function must be acyclic. The second condition, "internal consistency," requires that the outcome of a reduced game, say a, be always also the outcome of all reduced games that can be defined as follows: if a' is a predecessor of a, take the reduced game defined using as terminal nodes all those terminal nodes that defined the original game and are successors of a'. The third condition, called "subgame consistency," requires that if a is the outcome of a reduced game, then at each node leading to a, the player whose turn it is to move should choose, in the reduced game with terminal nodes a and the outcome that would be reached should he choose a strategy not leading to a, the strategy that leads to a. These conditions are shown to be not only necessary, but suffifient.

such an outcome function, the results of applying the concept of subgame-perfect equilibrium to this kind of problems are falsifiable.

These three works consider the behavioral principles behind the Nash equilibrium concept. In contrast, Chiappori (1988) considers the implications of cooperative behavior in a less abstract setting. Suppose that one observes a two-member household, each of whose members derives utility from his or her own consumption and leisure, as well as from the ones of the other member, facing a joint budget constraint. The question that Chiappori answers is: do there exist conditions on the observations of individual labor supply and aggregate consumption under which there exist individual preferences such that the observations are Pareto-efficient collective decisions, under these preferences and given (observed) individual wages? Chiappori shows that such conditions indeed exist,[9] and are both necessary and sufficient, and that there exist observations that could not be rationalized in the mentioned sense. It is concluded that the principles of Pareto behavior applied in this particular setting produce a falsifiable hypothesis.

The present paper is mostly related to Zhou (1999) and Sprumont (2000). The problem considered here is the following: suppose that one observes a finite set of players, each of whom chooses from an interval from the real line. A constraint is modeled as a lower and an upper bound to what a player can choose. Suppose that one observes a finite sequence of profiles of constraints and choices. In this paper I ask the question of what conditions on this sequence are necessary (and sufficient) for the existence of individual preferences over the Cartesian product of all intervals, such that for each observed profile of constraints, the corresponding observed profile of choices is a Nash equilibrium of the game played under the constrained domain if individuals have these preferences. (Trivial answers to the question are ruled out by assumptions explained below.)

The interest in continuous domains is easily motivated. Suppose that the government is trying to provide a public good and would like the consumers to reveal their valuations of it, or consider the cases of a private value auction, an economy in which externalities arise from the consumption or production of some commodity, or Bertrand or Cournot oligopolies. In all these cases, the set of conceivable actions of each player can be taken to be an interval in $\mathbb{R}$.

In that sense, my problem is clearly related to the one studied by Zhou (1999), and hence their differences, although simple, deserve to be pointed out. First, I do not assume that there are only two players. Second, I allow for constraints on the actions of the players to be exogenously imposed, which was not the case in Zhou (1999). Third, and most importantly, my requirement for the individual preferences is weaker that Zhou's in that I do not rule out the existence of other equilibria in the constrained games besides the ones that appear on the observed data set. This difference is crucial. In my case, the weaker requirement comes at the cost of needing to impose stronger restrictions

[9] The conditions, which are mediated by existencial quantifiers, amount to a generalization of the strong axiom of revealed preferences for collective choices.

on the class of preferences allowed, in order to rule out trivial results (in which players do not care about their own actions and, hence, every feasible outcome is Nash equilibrium) which would render the theory unfalsifiable. The return for incurring in this cost is that if an empirical application of my results is to be carried out, there is no need to argue that in the finite data set one has observed *all* the equilibria that the game has, which can be a very strong assumption.[10]

The assumption of continuous choice sets is an obvious difference with respect to Sprumont (2000) and Ray and Zhou (2000). Additionally, I impose weaker observational requirements on my tests than in both of these papers. I only require that for each observed profile of constraints, a profile of actions be observed. I do not require (actually, I do rule out) that all possible profiles of constraints be observed, as is explicitly required by Sprumont, and implicitly by Ray and Zhou, when they assume that the outcome of all reduced games is observable. Furthermore, I do not assume that all the equilibria of the constrained games have been observed, as explicitly does Sprumont and implicitly do Ray and Zhou, when they assume uniqueness of the subgame-perfect equilibria. In contrast to these two papers, this double nonexhaustiveness may come as an empirical advantage but, again, it implies my need to rule out triviality via strong assumptions.

Finally, obvious differences with the analysis of Ray and Zhou (2000) is that I study simultaneous-move games and with Chiappori (1988) that my main focus is noncooperative behavior.

As in the other papers on noncooperative behavior, in particular Zhou (1999) and Sprumont (2000), I take the principles of Nash behavior in their most salient instance. That is, I assume that individuals do the best for themselves given what the others are doing. I am hence subject to the general criticism to the Nash solution for its strong informational requirement (or extreme accuracy in conjecture formation) involved in individual decision-making processes. And in that sense, this work does not advance the theory started by Bernheim (1984) and Pearce (1984), where the question of rationalizability of strategic behavior is answered from the perspective of beliefs and not only of preferences, as I do here. The criticism is valid, but does not apply as strongly to my results as it does to general theoretical applications of Nash equilibrium, as will be argued in subsection 3.3.

3 Noncooperative behavior:

Let $\mathcal{I}$ be a nonempty, finite set of players. I will denote by $I \in \mathbb{N}$, $I \geqslant 2$, the number of players. Suppose that for each player $i \in \mathcal{I}$, the set of conceivable actions is the interval $A^i = [\underline{a}_i, \overline{a}_i]$, where $\underline{a}_i, \overline{a}_i \in \mathbb{R}$, $\underline{a}_i < \overline{a}_i$. This set A^i is

[10] Suppose that we reject the hypothesis of Nash behavior based on Zhou's test, using a finite time series of observations, on the premise that one more equilibrium would need to exist if all the observed outcomes also are to be equilibria. A "reasonable excuse" would be that the extra equilibrium just happens to not have been played, with which the empirical harshness of the test rapidly deteriorates.

player i's structural choice set, from where he could choose in the absence of exogenous constraints.[11]

Given these structural feasible sets, I want to consider the possibility that, conjunctionally, individuals are constrained in their choices. I model these constraints as lower and upper bounds to what they can choose. Besides these bounds, the only other information that I assume can be observed is actual individual choices. That is, I assume that a finite sequence is observed, each of whose elements specifies, for each player, a lower and an upper bound to what he can choose and an actual choice. Formally:

Definition 1 *A data set is a finite sequence*

$$\left(\left(a_{i,t}^{*},\underline{a}_{i,t},\overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)_{t=1}^{T}$$

such that:

$$\left(\forall t\in\{1,...,T\}\right)\left(\forall i\in\mathcal{I}\right):\left(\underline{a}_{i}\leqslant\underline{a}_{i,t}<\overline{a}_{i,t}\leqslant\overline{a}_{i}\wedge a_{i,t}^{*}\in\left[\underline{a}_{i,t},\overline{a}_{i,t}\right]\right)$$

Hence, for each observation $t\in\{1,...,T\}=\mathcal{T}$ and each player $i\in\mathcal{I}$, one observes three numbers: $\underline{a}_{i,t}$ and $\overline{a}_{i,t}$ are, respectively, the minimum and maximum values that player i can choose at observation t, whereas $a_{i,t}^{*}$ is what he actually chose.[12] Since one only wants to consider feasible data sets with meaningful constraints, the conditions that the conjunctional constraint be at least as tight that the structural one, without implying an empty or degenerate feasible set and that the actual choice be feasible are imposed by the definition.

My goal here is to derive testable implications on the observed sequence which are implied by rational behavior in the sense of Nash equilibrium. A definition of rational data set is then needed. I will say that a data set is, or, rather, can be rationalized by Nash behavior, if one can find individual preferences over the collective domain $\prod_{i\in\mathcal{I}}A^{i}$, such that for each observation $t\in\mathcal{T}$, the profile of choices $\left(a_{i,t}^{*}\right)_{i\in\mathcal{I}}$ is a Nash equilibrium of the game that these players would play, if they had these preferences and each one were constrained to choose from the subinterval $\left[\underline{a}_{i,t},\overline{a}_{i,t}\right]$. Of course, this definition is vacuous, and Nash behavior unfalsifiable, unless one restricts the class of preferences allowed in the rationalization. That is, if I allow preferences in which individuals are indifferent with respect to the value taken by their own choice, then every individual choice can be a best response, every outcome is therefore a Nash equilibrium and every data set is rationalizable. I then restrict the class of preferences, so as to imply that individuals always have unique best responses.

Before the formal definition of rationalizability, the following notational convention has to be introduced: given a player $i\in\mathcal{I}$, a function

$$U^{i}:A^{i}\times A^{-i}\longrightarrow\mathbb{R}$$

[11] Like the consumption set in consumer theory.

[12] Continuing the analogy of footnote 11, the interval $\left[\underline{a}_{i,t},\overline{a}_{i,t}\right]$ is like the budget set in consumer theory, whereas $a_{i,t}^{*}$ is analogous to the demand.

where $A^{-i} = \prod_{j\in\mathcal{I}\setminus\{i\}} A_j$, and vectors $\underline{a}', \overline{a}' \in \prod_{j\in\mathcal{I}} A_j$, $\underline{a}' \ll \overline{a}'$, I denote by $U^i_{\underline{a}',\overline{a}'}$ the restriction of U^i to

$$[\underline{a}'_i, \overline{a}'_i] \times \prod_{j\in\mathcal{I}\setminus\{i\}} [\underline{a}'_j, \overline{a}'_j]$$

Also, if each player $i \in \mathcal{I}$ is constrained to choose from a set $B^i \subseteq A^i$, and has preferences represented by

$$V^i : B^i \times \prod_{j\in\mathcal{I}\setminus\{i\}} B_j \longrightarrow \mathbb{R}$$

I denote by $N\left(\left(B^i, V^i\right)_{i\in\mathcal{I}}\right)$ the set of Nash equilibria of the game $\left(B^i, V^i\right)_{i\in\mathcal{I}}$.

With this notation, the definition of what data sets will be considered consistent with Nash behavior is:

Definition 2 *A data set*

$$\left(\left(a^*_{i,t}, \underline{a}_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)_{t=1}^{T}$$

is Nash-rationalizable if for each $i \in \mathcal{I}$ there exists

$$U^i : A^i \times A^{-i} \longrightarrow \mathbb{R}$$

continuous, such that for each $a_{-i} \in A^{-i}$, the function $U^i(\cdot, a_{-i})$ is differentiable and strongly concave, and for each $t \in \mathcal{T}$,

$$\left(a^*_{i,t}\right)_{i\in\mathcal{I}} \in N\left(\left(\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right], U^i_{\underline{a}_t,\overline{a}_t}\right)_{i\in\mathcal{I}}\right)$$

In this case, it is said that $\left(U^i\right)_{i\in\mathcal{I}}$ Nash-rationalizes the data.

The assumption that rules out trivial rationalizations is strong concavity in own actions, since it implies that individual best responses have to be unique, given the convexity of the constraint sets.[13] But I also impose other assumptions. Implicitly, I assume that preferences are representable by utility functions and that these are continuous. Continuity is imposed out of plausibility, in order to get some smoothness in the best responses of players, in the sense that small enough changes in actions by other players produce small changes in individual responses (by the theorem of the maximum). Since general representability results exist when preferences are continuous (Debreu, 1954) it is reasonable that these two assumptions come together.[14] Neither the assumption of concavity

[13]For the purposes of this and the next sections, strong quasiconcavity would suffice. Actual concavity is used only in section 5.1, but the assumption is introduced here for the sake of consistency.

[14]I also assume that these utility functions are differentiable in own action, but this assumption plays no role in the general result, and we could dispense with it for the purposes of this section. The assumption will be used in subsection 5.1, when I derive further testable restrictions under extra assumptions.

in own action nor the one of continuity are extraneous to game theory, as they were used in the proof of existence of Nash equilibrium by Debreu (1952).[15] Notice also that I do not impose any monotonicity requirements. The reason is that imposing increasing monotonicity would trivialize the results, as for each player $i \in \mathcal{I}$, $a_{i,t}^* = \overline{a}_{i,t}$ would be a dominant strategy, while the same would happen, with $a_{i,t}^* = \underline{a}_{i,t}$, if decreasing monotonicity were imposed.

The reason why I choose a weak definition of rationalization, in the sense that it only requires that for each $t \in \mathcal{T}$,

$$\left(a_{i,t}^*\right)_{i\in\mathcal{I}} \in N\left(\left(\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right], U_{\underline{a}_t,\overline{a}_t}^i\right)_{i\in\mathcal{I}}\right)$$

and not that

$$N\left(\left(\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right], U_{\underline{a}_t,\overline{a}_t}^i\right)_{i\in\mathcal{I}}\right) = \left\{\left(a_{i,t}^*\right)_{i\in\mathcal{I}}\right\}$$

is that I do not want to assume, or imply, that one observes all possible equilibria of the constrained games. Only one equilibrium is assumed to be observed, and there is no reason for this equilibrium to be unique. Moreover, under my assumptions it could very well be the case that for $t, t' \in \mathcal{T}$ we have $\left(a_{i,t}^*\right)_{i\in\mathcal{I}} \neq \left(a_{i,t'}^*\right)_{i\in\mathcal{I}}$ and still

$$\left(a_{i,t'}^*\right)_{i\in\mathcal{I}} \in N\left(\left(\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right], U_{\underline{a}_t,\overline{a}_t}^i\right)_{i\in\mathcal{I}}\right)$$

This is just a property of Nash-equilibrium in this context: suppose that for some $\underline{a}', \overline{a}' \in \prod_{i\in\mathcal{I}} A^i$, $\underline{a}' \ll \overline{a}'$, and some $(a_i^*)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} (\underline{a}_i', \overline{a}_i')$, one has that

$$(a_i^*)_{i\in\mathcal{I}} \in N\left(\left(\left[\underline{a}_i', \overline{a}_i'\right], U_{\underline{a}',\overline{a}'}^i\right)_{i\in\mathcal{I}}\right)$$

Then, for all $\underline{a}'', \overline{a}'' \in \prod_{i\in\mathcal{I}} A^i$, $\underline{a}'' \ll \overline{a}''$,

$$(a_i^*)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \left[\underline{a}_i'', \overline{a}_i''\right] \Longrightarrow (a_i^*)_{i\in\mathcal{I}} \in N\left(\left(\left[\underline{a}_i'', \overline{a}_i''\right], U_{\underline{a}'',\overline{a}''}^i\right)_{i\in\mathcal{I}}\right)$$

because

$$(a_i^*)_{i\in\mathcal{I}} \in N\left(\left(\left[\underline{a}_i', \overline{a}_i'\right], U_{\underline{a}',\overline{a}'}^i\right)_{i\in\mathcal{I}}\right)$$

implies that for each $i \in \mathcal{I}$, a_i^* is a local maximizer of $U^i\left(\cdot, a_{-i}^*\right)$ over $[\underline{a}_i', \overline{a}_i']$, and then, since $a_i^* \in (\underline{a}_i', \overline{a}_i')$, it follows from the concavity assumption that a_i^* is global maximizer of $U^i\left(\cdot, a_{-i}^*\right)$ over $[\underline{a}_i, \overline{a}_i]$.

Hence, in order to maintain my weak observational requirements, I use a weak concept of rationalization. This approach is not new, as it is the one taken, for example, by Brown and Matzkin (1996). There, a finite data set of prices and endowments is said to be rationalizable if there exist preferences of

[15] Debreu only requires weak concavity, as does Zhou (1999). They both impose continuity.

the agents such that each observation of prices is a competitive equilibrium price vector of the exchange economy given the corresponding endowments. It is not required that such an equilibrium be unique, nor is there a reason to expect that it will be. I will address this issue again in subsection 3.2.

There are two reasons why I chose to rationalize a finite data set, rather than some outcome correspondence. (These results must be seen as nonparametric.) One is that finite data sets is what typically one will have available when trying to apply the results obtained here. The other reason is deeper: since parametric functions typically are derived from finite data, rejection of the rationalizability hypothesis could come from either a nonrationalizable data set or a nonrationalizable functional form applied to probably rationalizable data (Varian (1983) and Chiappori (1988)). This does not mean that the parametric approach is not interesting, but rather that its power is fully exploited after a nonparametric test.

3.1 General testable restrictions:

Following the Popperian postulate, I now study the problem of what conditions must the observables of the theory satisfy, if one is to say that they are the result of the behavior assumed by the theory. In other words, if individuals behave according to the principles behind Nash equilibrium, what are the necessary conditions (which should not be tautologies,) that have to be exhibited by the observed data set? Theorem 1 below derives these conditions and shows that they are all the conditions that can be derived, as they are independent of one another and also sufficient.

Theorem 1 *A data set*

$$\left(\left(a^*_{i,t}, \underline{a}_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)_{t=1}^{T}$$

is Nash-rationalizable if, and only if, for all $t, t' \in \mathcal{T}$ *and all* $i \in \mathcal{I}$:

1.
$$\left.\begin{array}{c} a^*_{-i,t} = a^*_{-i,t'} \\ a^*_{i,t} \in \left[\underline{a}_{i,t'}, \overline{a}_{i,t'}\right] \\ a^*_{i,t'} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right] \end{array}\right\} \Longrightarrow a^*_{i,t} = a^*_{i,t'}$$

2.
$$\left.\begin{array}{c} a^*_{-i,t} = a^*_{-i,t'} \\ a^*_{i,t} \in \left(\underline{a}_{i,t}, \overline{a}_{i,t}\right] \\ a^*_{i,t} \geqslant \overline{a}_{i,t'} \end{array}\right\} \Longrightarrow a^*_{i,t'} = \overline{a}_{i,t'}$$

3.
$$\left.\begin{array}{c} a^*_{-i,t} = a^*_{-i,t'} \\ a^*_{i,t} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right) \\ a^*_{i,t} \leqslant \underline{a}_{i,t'} \end{array}\right\} \Longrightarrow a^*_{i,t'} = \underline{a}_{i,t'}$$

Proof. Necessity: Let $\left(U^i\right)_{i\in\mathcal{I}}$ Nash-rationalize the data.

Suppose that $\exists i\in\mathcal{I}$ and $\exists t,t'\in\mathcal{T}$ such that $a^*_{-i,t}=a^*_{-i,t'}$, $a^*_{i,t}\in\left[\underline{a}_{i,t'},\overline{a}_{i,t'}\right]$, $a^*_{i,t'}\in\left[\underline{a}_{i,t},\overline{a}_{i,t}\right]$ and $a^*_{i,t}\neq a^*_{i,t'}$. Without loss of generality, assume that $U^i\left(a^*_{i,t},a^*_{-i,t}\right)\leqslant U^i\left(a^*_{i,t'},a^*_{-i,t'}\right)$. Let $a_i=\frac{1}{2}\left(a^*_{i,t}+a^*_{i,t'}\right)$. Clearly, $a_i\in\left[\underline{a}_{i,t},\overline{a}_{i,t}\right]$, whereas by strong concavity $U^i\left(a_i,a^*_{-i,t}\right)>U^i\left(a^*_{i,t},a^*_{-i,t}\right)$, contradicting the fact that $\left(U^i\right)_{i\in\mathcal{I}}$ Nash-rationalizes the data set. This proves condition (1).

For condition (2), let $i\in\mathcal{I}$ and $t,t'\in\mathcal{T}$ be such that $a^*_{-i,t}=a^*_{-i,t'}$, $a^*_{i,t}\in\left(\underline{a}_{i,t},\overline{a}_{i,t}\right]$, $a^*_{i,t}\geqslant\overline{a}_{i,t'}$. I first claim that $\forall a\in\left[\underline{a}_i,a^*_{i,t}\right)$, $U^i\left(a,a^*_{-i,t}\right)<U^i\left(a^*_{i,t},a^*_{-i,t}\right)$. To see this, suppose not: $\exists a\in\left[\underline{a}_i,a^*_{i,t}\right)$: $U^i\left(a,a^*_{-i,t}\right)\geqslant U^i\left(a^*_{i,t},a^*_{-i,t}\right)$. If $a\geqslant\underline{a}_{i,t}$, then

$$\frac{1}{2}a+\frac{1}{2}a^*_{i,t}\in\left[\underline{a}_{i,t},\overline{a}_{i,t}\right]$$

and

$$\begin{aligned}U^i\left(\frac{1}{2}a+\frac{1}{2}a^*_{i,t},a^*_{-i,t}\right) &> \frac{1}{2}U^i\left(a,a^*_{-i,t}\right)+\frac{1}{2}U^i\left(a^*_{i,t},a^*_{-i,t}\right)\\ &\geqslant U^i\left(a^*_{i,t},a^*_{-i,t}\right)\end{aligned}$$

Hence, it follows from rationalizability that $a<\underline{a}_{i,t}$. Let

$$\theta=\frac{a^*_{i,t}-\underline{a}_{i,t}}{2\left(a^*_{i,t}-a\right)}$$

Clearly, $\theta\in(0,1)$ and, therefore, by strong concavity,

$$\begin{aligned}U^i\left(\theta a+(1-\theta)\,a^*_{i,t},a^*_{-i,t}\right) &> \theta U^i\left(a,a^*_{-i,t}\right)+(1-\theta)\,U^i\left(a^*_{i,t},a^*_{-i,t}\right)\\ &\geqslant U^i\left(a^*_{i,t},a^*_{-i,t}\right)\end{aligned}$$

However,

$$\begin{aligned}\theta a+(1-\theta)\,a^*_{i,t} &= \frac{a^*_{i,t}-\underline{a}_{i,t}}{2\left(a^*_{i,t}-a\right)}a+\left(1-\frac{a^*_{i,t}-\underline{a}_{i,t}}{2\left(a^*_{i,t}-a\right)}\right)a^*_{i,t}\\ &= \frac{a^*_{i,t}-\underline{a}_{i,t}}{2\left(a^*_{i,t}-a\right)}a+\frac{a^*_{i,t}-2a+\underline{a}_{i,t}}{2\left(a^*_{i,t}-a\right)}a^*_{i,t}\\ &= \frac{aa^*_{i,t}-a\underline{a}_{i,t}+\left(a^*_{i,t}\right)^2-2aa^*_{i,t}+\underline{a}_{i,t}a^*_{i,t}}{2\left(a^*_{i,t}-a\right)}\\ &= \frac{\left(a^*_{i,t}\right)^2-aa^*_{i,t}-a\underline{a}_{i,t}+\underline{a}_{i,t}a^*_{i,t}}{2\left(a^*_{i,t}-a\right)}\\ &= \frac{a^*_{i,t}\left(a^*_{i,t}-a\right)+\underline{a}_{i,t}\left(a^*_{i,t}-a\right)}{2\left(a^*_{i,t}-a\right)}\\ &= \frac{a^*_{i,t}+\underline{a}_{i,t}}{2}\end{aligned}$$

which implies that $\theta a+(1-\theta)\,a^*_{i,t}\in\left[\underline{a}_{i,t},\overline{a}_{i,t}\right]$. This contradicts the fact that $\left(U^i\right)_{i\in\mathcal{I}}$ Nash-rationalizes the data set.

I also claim that $\forall a,a'\in\left[\underline{a}_i,a^*_{i,t}\right]$, such that $a<a'$, $U^i\left(a,a^*_{-i,t}\right)<U^i\left(a',a^*_{-i,t}\right)$. If $a'=a^*_{i,t}$, the result follows from the the previous claim. Now, suppose that $a'<a^*_{i,t}$. Let

$$\theta=\frac{a^*_{i,t}-a'}{a^*_{i,t}-a}$$

Clearly, $\theta\in(0,1)$ and therefore

$$\begin{aligned} U^i\left(\theta a+(1-\theta)\,a^*_{i,t},a^*_{-i,t}\right) &> \theta U^i\left(a,a^*_{-i,t}\right)+(1-\theta)\,U^i\left(a^*_{i,t},a^*_{-i,t}\right)\\ &> U^i\left(a,a^*_{-i,t}\right)\end{aligned}$$

where the second inequality also follows from the previous claim. Now,

$$\begin{aligned}\theta a+(1-\theta)\,a^*_{i,t} &= \frac{a^*_{i,t}-a'}{a^*_{i,t}-a}a+\left(1-\frac{a^*_{i,t}-a'}{a^*_{i,t}-a}\right)a^*_{i,t}\\ &= \frac{a^*_{i,t}-a'}{a^*_{i,t}-a}a+\frac{a'-a}{a^*_{i,t}-a}a^*_{i,t}\\ &= \frac{aa^*_{i,t}-aa'+a'a^*_{i,t}-aa^*_{i,t}}{a^*_{i,t}-a}\\ &= a'\end{aligned}$$

which establishes the claim.

Now, since $a^*_{-i,t}=a^*_{-i,t'}$, it is clear $\forall a,a'\in\left[\underline{a}_i,a^*_{i,t}\right]$, such that $a<a'$, $U^i\left(a,a^*_{-i,t'}\right)<U^i\left(a',a^*_{-i,t'}\right)$. Then, since $a^*_{i,t}\geqslant\overline{a}_{i,t'}$ it follows that $\forall a,a'\in[\underline{a}_i,\overline{a}_{i,t'}]$, such that $a<a'$, $U^i\left(a,a^*_{-i,t'}\right)<U^i\left(a',a^*_{-i,t'}\right)$ from where, by rationalizability, $a^*_{i,t'}=\overline{a}_{i,t'}$

Condition (3) can be argued in a similar way.

Sufficiency: For each $i\in\mathcal{I}$, construct $\mathcal{T}^i\subseteq\mathcal{T}$ according to the following algorithm.

Algorithm 1 *Input:*

$$\left(\left(a^*_{i,t},\underline{a}_{i,t},\overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)^T_{t=1}$$

1. $\mathcal{S}=\mathcal{T}$, $\mathcal{T}^i=\varnothing$
2. $t=\min\mathcal{S}$
3. $\Gamma=\left\{t'\in\mathcal{T}|\,a^*_{-i,t'}=a^*_{-i,t}\right\}$
4. $\Delta=\left\{t'\in\Gamma|\,a^*_{i,t'}\in\left(\underline{a}_{i,t'},\overline{a}_{i,t'}\right)\right\}$
5. *If* $\Delta\neq\varnothing$, *then let* $\tau=\min\left(Arg\min_{t'\in\Delta}a^*_{i,t'}\right)$ *and go to 9.*

6. $\Theta = \left\{ t' \in \Gamma | \, a^*_{i,t'} = \overline{a}_{i,t'} \right\}$

7. *If* $\Theta \neq \varnothing$, *then let* $\tau = \min \left(Arg \max_{t' \in \Theta} a^*_{i,t'} \right)$ *and go to 9.*

8. $\tau = \min \left(Arg \min_{t' \in \Gamma} a^*_{i,t'} \right)$

9. $\mathcal{T}^i = \mathcal{T}^i \cup \{\tau\}$

10. $\mathcal{S} = \mathcal{S} \backslash \Gamma$

11. *If* $\mathcal{S} = \varnothing$, *stop.*

12. *Go to 2.*

Output: $\mathcal{T}^i \subseteq \mathcal{T}$

The output of the algorithm has the following two properties:

$$\begin{aligned} (\forall t \in \mathcal{T}) \left(\exists \tau \in \mathcal{T}^i \right) &\quad : \quad a^*_{-i,\tau} = a^*_{-i,t} \\ \left(\forall \tau, \tau' \in \mathcal{T}^i : \tau \neq \tau' \right) &\quad : \quad a^*_{-i,\tau} \neq a^*_{-i,\tau'} \end{aligned}$$

which imply that one can construct the following (well-defined) function. Let $\phi^i : \left\{ a^*_{-i,t'} \right\}_{t' \in \mathcal{T}} \longrightarrow A^i$ be defined by

$$\phi^i \left(a^*_{-i,t'} \right) = a^*_{i,\tau_t} \text{ where } \tau_t \in \mathcal{T}^i \text{ is such that } a^*_{-i,\tau_t} = a^*_{-i,t}$$

Since $\left\{ a^*_{-i,t'} \right\}_{t' \in \mathcal{T}} \subseteq A^{-i}$ is closed and $\phi^i : \left\{ a^*_{-i,t'} \right\}_{t' \in \mathcal{T}} \longrightarrow A^i$ is continuous and bounded, by Tietze´s extension theorem (see, for example, theorem 3.12.3 in Bridges, 1988) there exists $\Phi^i : A^{-i} \longrightarrow A^i$, continuous, such that

$$\left(\forall a_{-i} \in \left\{ a^*_{-i,t'} \right\}_{t' \in \mathcal{T}} \right) : \Phi^i \left(a_{-i} \right) = \phi^i \left(a_{-i} \right)$$

Fix one such $\Phi^i : A^{-i} \longrightarrow A^i$ and define $U^i : A^i \times A^{-i} \longrightarrow \mathbb{R}$ as

$$U^i \left(a_i, a_{-i} \right) = - \left(a_i - \Phi^i \left(a_{-i} \right) \right)^2$$

That U^i is continuous and $\forall a_{-i} \in A^{-i}$, $U^i \left(\cdot, a_{-i} \right)$ is differentiable and strongly concave is straightforward. Hence, all that remains to show is that $\left(U^i \right)_{i \in \mathcal{I}}$ rationalizes the data set:

Let $i \in \mathcal{I}$ and $t \in \mathcal{T}$. Define $\Gamma_{i,t} = \left\{ t' \in \mathcal{T} | \, a^*_{-i,t'} = a^*_{-i,t} \right\}$. One of the following three mutually exclusive cases must hold:

Case 1: $\left(\exists t' \in \Gamma_{i,t} \right) : a^*_{i,t'} \in \left(\underline{a}_{i,t'}, \overline{a}_{i,t'} \right)$

Case 2: $\left(\left(\forall t' \in \Gamma_{i,t} \right) : a^*_{i,t'} = \underline{a}_{i,t'} \vee a^*_{i,t'} = \overline{a}_{i,t'} \right) \wedge \left(\left(\exists t' \in \Gamma_{i,t} \right) : a^*_{i,t'} = \overline{a}_{i,t'} \right)$

Case 3: $\left(\forall t' \in \Gamma_{i,t} \right) : a^*_{i,t'} = \underline{a}_{i,t'}$

Suppose that case 1 holds. Let

$$t'_t = \min \left(Arg \min_{t' \in \Gamma_{i,t} : a^*_{i,t'} \in \left(\underline{a}_{i,t'}, \overline{a}_{i,t'} \right)} a^*_{i,t'} \right)$$

By construction, $a^*_{-i,t'_t} = a^*_{-i,t}$ and

$$\begin{aligned}\Phi^i\left(a^*_{-i,t}\right) &= \Phi^i\left(a^*_{-i,t'_t}\right)\\ &= \phi^i\left(a^*_{-i,t'_t}\right)\\ &= a^*_{i,t'_t}\end{aligned}$$

from where, if $t'_t = t$,

$$\begin{aligned}a^*_{i,t} &= \arg\max_{a_i\in A^i} U^i\left(a_i,a^*_{-i,t}\right)\\ &= \arg\max_{a_i\in\left[\underline{a}_{i,t},\overline{a}_{i,t}\right]} U^i\left(a_i,a^*_{-i,t}\right)\end{aligned}$$

Alternatively, suppose that $t \in \mathcal{T}\setminus\{t'_t\}$.

If $a^*_{i,t} \in \left(\underline{a}_{i,t},\overline{a}_{i,t}\right)$, then by, condition (2), $a^*_{i,t'_t} < \overline{a}_{i,t}$ and $a^*_{i,t} < \overline{a}_{i,t'_t}$, whereas, by condition (3), $a^*_{i,t'_t} > \underline{a}_{i,t}$ and $a^*_{i,t} > \underline{a}_{i,t'_t}$. Then, by condition (1), $a^*_{i,t} = a^*_{i,t'_t}$, and, therefore,

$$\begin{aligned}a^*_{i,t} &= \arg\max_{a_i\in A^i} U^i\left(a_i,a^*_{-i,t}\right)\\ &= \arg\max_{a_i\in\left[\underline{a}_{i,t},\overline{a}_{i,t}\right]} U^i\left(a_i,a^*_{-i,t}\right)\end{aligned}$$

On the other hand, suppose that $a^*_{i,t} = \overline{a}_{i,t}$. If $\overline{a}_{i,t} \leqslant a^*_{i,t'_t}$ it is obvious that

$$\begin{aligned}\arg\max_{a_i\in\left[\underline{a}_{i,t},\overline{a}_{i,t}\right]} U^i\left(a_i,a^*_{-i,t}\right) &= \overline{a}_{i,t}\\ &= a^*_{i,t}\end{aligned}$$

whereas if $\overline{a}_{i,t} > a^*_{i,t'_t}$, there are four possibilities: (i) if $\underline{a}_{i,t} < a^*_{i,t'_t}$ and $\overline{a}_{i,t'_t} \leqslant \overline{a}_{i,t}$, then $a^*_{i,t} \in \left(\underline{a}_{i,t},\overline{a}_{i,t}\right]$ and $\overline{a}_{i,t'_t} \leqslant a^*_{i,t}$, which implies, by condition (2), that $a^*_{i,t'_t} = \overline{a}_{i,t'_t}$ and contradicts the fact that $a^*_{i,t'_t} \in \left(\underline{a}_{i,t'_t},\overline{a}_{i,t'_t}\right)$; (ii) If $\underline{a}_{i,t} < a^*_{i,t'_t}$, $\overline{a}_{i,t'_t} > \overline{a}_{i,t}$ and $\underline{a}_{i,t'_t} \leqslant a^*_{i,t}$, then $a^*_{i,t'_t} \in \left[\underline{a}_{i,t},\overline{a}_{i,t}\right]$ and $a^*_{i,t} \in \left[\underline{a}_{i,t'_t},\overline{a}_{i,t'_t}\right]$ and, therefore, by condition (1), $a^*_{i,t} = a^*_{i,t'_t}$, which implies that

$$\begin{aligned}a^*_{i,t} &= \arg\max_{a_i\in A^i} U^i\left(a_i,a^*_{-i,t}\right)\\ &= \arg\max_{a_i\in\left[\underline{a}_{i,t},\overline{a}_{i,t}\right]} U^i\left(a_i,a^*_{-i,t}\right)\end{aligned}$$

(iii) If $\underline{a}_{i,t} < a^*_{i,t'_t}$, $\overline{a}_{i,t'_t} > \overline{a}_{i,t}$ and $\underline{a}_{i,t'_t} > a^*_{i,t}$, then

$$\begin{aligned}\overline{a}_{i,t} &= a^*_{i,t}\\ &< \underline{a}_{i,t'_t}\\ &< a^*_{i,t'_t}\\ &< \overline{a}_{i,t}\end{aligned}$$

which is an obvious contradiction; (iv) finally, if $\underline{a}_{i,t} \geqslant a^*_{i,t'_t}$, then $a^*_{i,t'_t} \in \left[\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t}\right)$ and $\underline{a}_{i,t} \geqslant a^*_{i,t'_t}$ imply, by condition (3), that $a^*_{i,t} = \underline{a}_{i,t}$, which contradicts the fact that $a^*_{i,t} = \overline{a}_{i,t} > \underline{a}_{i,t}$.

Finally, suppose that $a^*_{i,t} = \underline{a}_{i,t}$. If $a^*_{i,t'_t} \leqslant \underline{a}_{i,t}$ it is obvious that

$$a^*_{i,t} = \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right)$$

whereas if $a^*_{i,t'_t} > \underline{a}_{i,t}$, there are three possibilities: (i) if $\overline{a}_{i,t} \leqslant a^*_{i,t'_t}$, then $a^*_{i,t'_t} \in \left(\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t}\right]$ and $\overline{a}_{i,t} \leqslant a^*_{i,t'_t}$ imply, by condition (2), that $a^*_{i,t} = \overline{a}_{i,t}$, which contradicts the fact that $a^*_{i,t} = \underline{a}_{i,t} < \overline{a}_{i,t}$, (ii) if $\overline{a}_{i,t} > a^*_{i,t'_t}$ and $\underline{a}_{i,t'_t} \geqslant a^*_{i,t}$, then $a^*_{i,t} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right)$ and $a^*_{i,t} \leqslant \underline{a}_{i,t'_t}$ imply, by condition (3), that $a^*_{i,t'_t} = \underline{a}_{i,t'_t}$, which contradicts the fact that $a^*_{i,t'_t} \in \left(\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t}\right)$; (iii) finally, if $\overline{a}_{i,t} > a^*_{i,t'_t}$ and $\underline{a}_{i,t'_t} < a^*_{i,t}$, then $a^*_{i,t'_t} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]$ and $a^*_{i,t} \in \left[\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t}\right]$ and, therefore, by condition (1), $a^*_{i,t} = a^*_{i,t'_t}$, which implies that

$$\begin{array}{rcl} a^*_{i,t} & = & \underline{a}_{i,t} \\ & < & a^*_{i,t'} \\ & = & a^*_{i,t} \end{array}$$

which is an obvious contradiction.

Suppose now that case 2 holds. Let

$$t'_t = \min\left(Arg \max_{t' \in \Gamma_{i,t} : a^*_{i,t'} = \overline{a}_{i,t'}} a^*_{i,t'}\right)$$

By construction, $a^*_{-i,t'_t} = a^*_{-i,t}$ and

$$\begin{array}{rcl} \Phi^i\left(a^*_{-i,t}\right) & = & \Phi^i\left(a^*_{-i,t'_t}\right) \\ & = & \phi^i\left(a^*_{-i,t'_t}\right) \\ & = & a^*_{i,t'_t} \end{array}$$

from where, if $t'_t = t$,

$$\begin{array}{rcl} a^*_{i,t,} & = & \arg \max_{a_i \in A^i} U^i\left(a_i, a^*_{-i,t}\right) \\ & = & \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right) \end{array}$$

Now, suppose that $t \in \mathcal{T} \setminus \{t'_t\}$.

If $\overline{a}_{i,t} \leqslant a^*_{i,t'_t}$, then, by condition (3), $a^*_{i,t} = \overline{a}_{i,t}$, whereas by construction

$$\overline{a}_{i,t} = \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right)$$

Alternatively, $\overline{a}_{i,t} > a^*_{i,t'_t}$, from where, by construction, $a^*_{i,t} = \underline{a}_{i,t}$. There are three possibilities: (i) If $\underline{a}_{i,t} \leqslant \underline{a}_{i,t'_t}$, then, by condition (3), $a^*_{i,t'_t} = \underline{a}_{i,t'_t} < \overline{a}_{i,t'_t}$, which is a contradiction, because $a^*_{i,t'_t} = \overline{a}_{i,t'_t}$; (ii) if $\underline{a}_{i,t'_t} < \underline{a}_{i,t} \leqslant a^*_{i,t'_t}$, then $\underline{a}_{i,t} \leqslant a^*_{i,t'_t} \leqslant \overline{a}_{i,t}$ and $\underline{a}_{i,t'_t} < \underline{a}_{i,t} = a^*_{i,t} \leqslant a^*_{i,t'_t} = \overline{a}_{i,t'_t}$, which implies, by condition (1), that $a^*_{i,t} = a^*_{i,t'_t}$ and hence that

$$\begin{aligned} a^*_{i,t} &= \arg\max_{a_i \in A^i} U^i\left(a_i, a^*_{-i,t}\right) \\ &= \arg\max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right) \end{aligned}$$

(iii) if $a^*_{i,t'_t} < \underline{a}_{i,t} = a^*_{i,t}$, then, by construction,

$$a^*_{i,t} = \arg\max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right)$$

Finally, suppose that case 3 holds. Let

$$t'_t = \min\left(Arg \min_{t' \in \Gamma_{i,t}} a^*_{i,t'}\right)$$

By construction, $a^*_{-i,t'_t} = a^*_{-i,t}$ and

$$\begin{aligned} \Phi^i\left(a^*_{-i,t}\right) &= \Phi^i\left(a^*_{-i,t'_t}\right) \\ &= \phi^i\left(a^*_{-i,t'_t}\right) \\ &= a^*_{i,t'_t} \end{aligned}$$

Since $a^*_{i,t} = \underline{a}_{i,t} \geqslant a^*_{i,t'_t}$, it follows that

$$a^*_{i,t} = \arg\max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right)$$

■

The necessity of the conditions of the theorem, and the fact that they are not tautologies, as it is easy to see that there exist data sets that violate them, imply that the hypothesis of Nash behavior under the assumptions made here is falsifiable. It is also easy to see that the conditions are independent of one another. What is more important is that their sufficiency implies that the hypothesis does not have further or stronger testable restrictions.[16]

The intuition for the conditions of the theorem is simple. The first condition is a straightforward restatement of the Weak Axiom of Revealed Preferences,

[16] Let **H** be a hypothesis and let **T** be a testable restriction implied by it (that is, $\mathbf{H} \Longrightarrow \mathbf{T}$, where **T** involves only observables of the theory). A stronger testable restriction would be a condition **ST** on observables only, such that $\mathbf{H} \Longrightarrow \mathbf{ST}$, but $\neg(\mathbf{T} \Longrightarrow \mathbf{ST})$. However, if $\mathbf{T} \Longrightarrow \mathbf{H}$, it is clear that stronger testable restrictions cannot exist.

for each $i \in \mathcal{I}$ and for pairs $t, t' \in \mathcal{T}$ such that $a^*_{-i,t} = a^*_{-i,t'}$:

$$a^*_{i,t'} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right] \Longrightarrow \left\{ \begin{array}{c} a^*_{i,t} = a^*_{i,t'} \\ \text{or} \\ a^*_{i,t} \notin \left[\underline{a}_{i,t'}, \overline{a}_{i,t'}\right] \end{array} \right.$$

Whenever a^*_{-i} is fixed, actions of player i, under the hypothesis of Nash behavior, ought to maximize a fixed utility function $U^i\left(\cdot, a^*_{-i}\right)$ subject to the particular constraints. The Weak Axiom of Revealed Preference is a well known necessary condition for this behavior, as shown in Richter (1966).

The second and third conditions are axioms of revealed monotonicity: when player i chooses $a^*_{i,t} \in \left(\underline{a}_{i,t}, \overline{a}_{i,t}\right]$, over $\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]$, given $a^*_{-i,t}$, by strong concavity he is revealing that $U^i\left(\cdot, a^*_{-i,t}\right)$ is strictly increasing on $\left[\underline{a}_i, a^*_{i,t}\right]$. Hence, conditional on $a^*_{-i,t'} = a^*_{-i,t}$, if $\overline{a}_{i,t'} \leqslant a^*_{i,t}$, then over $\left[\underline{a}_{i,t'}, \overline{a}_{i,t'}\right]$ player i ought to choose $a^*_{i,t'} = \overline{a}_{i,t'}$. Similarly, when player i chooses $a^*_{i,t} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right)$, over $\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]$, given $a^*_{-i,t}$, by strong concavity he is revealing that $U^i\left(\cdot, a^*_{-i,t}\right)$ is strictly decreasing on $\left[a^*_{i,t}, \overline{a}_i\right]$. Hence, conditional on $a^*_{-i,t'} = a^*_{-i,t}$, if $\underline{a}_{i,t'} \geqslant a^*_{i,t}$, then over $\left[\underline{a}_{i,t'}, \overline{a}_{i,t'}\right]$ player i ought to choose $a^*_{i,t'} = \underline{a}_{i,t'}$.

3.2 Weakness of the rationalization:

Recall that my definition of Nash-rationalizability is weak in the sense that it only requires that each observed choice be an element of the Nash set of the corresponding restricted game. As mentioned before, I do not (nor do I want to) require that the Nash set of the restricted game be the singleton set containing the observed choice. The risk that I am taking is then that, even though I have imposed conditions to rule out the result that every possible outcome is a Nash equilibrium, it could still be the case that almost every possible outcome is a Nash equilibrium, and with the rationalization of theorem 1 one happens to, in particular, pick the observed ones. As the following theorem shows, the conditions of theorem 1 allow for local uniqueness of the observed Nash equilibria (a property that will be generically shared by the rationalizations of Brown and Matzkin (1996), *mutatis mutandis*).

Theorem 2 *If a data set*

$$\left(\left(a^*_{i,t}, \underline{a}_{i,t}, \overline{a}_{i,t}\right)_{i \in \mathcal{I}}\right)_{t=1}^{T}$$

satisfies that for every $t, t' \in \mathcal{T}$ and every $i \in \mathcal{I}$:

1.

$$\left. \begin{array}{c} a^*_{-i,t} = a^*_{-i,t'} \\ a^*_{i,t} \in \left[\underline{a}_{i,t'}, \overline{a}_{i,t'}\right] \\ a^*_{i,t'} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right] \end{array} \right\} \Longrightarrow a^*_{i,t} = a^*_{i,t'}$$

2.

$$\left.\begin{array}{c} a^*_{-i,t} = a^*_{-i,t'} \\ a^*_{i,t} \in \left(\underline{a}_{i,t}, \overline{a}_{i,t}\right] \\ a^*_{i,t} \geqslant \overline{a}_{i,t'} \end{array}\right\} \implies a^*_{i,t'} = \overline{a}_{i,t'}$$

3.

$$\left.\begin{array}{c} a^*_{-i,t} = a^*_{-i,t'} \\ a^*_{i,t} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right) \\ a^*_{i,t} \leqslant \underline{a}_{i,t'} \end{array}\right\} \implies a^*_{i,t'} = \underline{a}_{i,t'}$$

then, for each $i \in \mathcal{I}$ there exists $U^i : A^i \times A^{-i} \longrightarrow \mathbb{R}$, such that $\left(U^i\right)_{i\in\mathcal{I}}$ Nash-rationalizes the data set with the following property: for each $t \in \mathcal{T}$, there exists an open neighborhood of $\left(a^*_{i,t}\right)_{i\in\mathcal{I}}$, $\mathcal{O}\left(\left(a^*_{i,t}\right)_{i\in\mathcal{I}}\right)$, such that:

$$N\left(\left(\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right], U^i_{\overline{a}_t}\right)_{i\in\mathcal{I}}\right) \cap \mathcal{O}\left(\left(a^*_{i,t}\right)_{i\in\mathcal{I}}\right) = \left\{\left(a^*_{i,t}\right)_{i\in\mathcal{I}}\right\}$$

Proof. For each $i \in \mathcal{I}$, let $\mathcal{T}^i$ be defined using algorithm 1 and define

$$\overline{\varepsilon}_i = \min_{t,t' \in \mathcal{T}^i : t \neq t'} \left\{\left\|a^*_{-i,t} - a^*_{-i,t'}\right\|\right\}$$

Since $\#\mathcal{T}^i \leqslant T < \infty$ and $\forall t, t' \in \mathcal{T}^i$, $t \neq t'$, $a^*_{-i,t} \neq a^*_{-i,t'}$, it follows that $\overline{\varepsilon}_i > 0$. Let $\varepsilon_i \in (0, \overline{\varepsilon}_i)$ and define the set

$$C^i = \left(\bigcup_{t\in\mathcal{T}^i} \overline{B_{\frac{\varepsilon_i}{2}}\left(a^*_{-i,t}\right)}\right) \cap A^{-i}$$

and the function $\phi^i : C^i \longrightarrow A^i$ by

$$\phi^i\left(a_{-i}\right) = \min\left\{Arg \max_{a_i \in \left\{a^*_{i,t}\right\}_{t\in\mathcal{T}^i}} \left(\min_{t'\in\mathcal{T}^i}\left(\left(\left\|a_{-i} - a^*_{-i,t'}\right\| - \frac{\varepsilon_i}{2}\right)\left(a_i - a^*_{i,t'}\right)^2\right)\right)\right\}$$

I first show that

$$\left(\forall t \in \mathcal{T}^i\right)\left(\forall a_{-i} \in \overline{B_{\frac{\varepsilon_i}{2}}\left(a^*_{-i,t}\right)} \cap A^{-i}\right) : \phi^i\left(a_{-i}\right) = a^*_{i,t}$$

To see this, let $t \in \mathcal{T}^i$ be fixed and fix a_{-i} in the relevant subset of A^{-i}. Consider, for each $a_i \in \left\{a^*_{i,t''}\right\}_{t''\in\mathcal{T}^i}$, the problem

$$\min_{t'\in\mathcal{T}^i}\left(\left(\left\|a_{-i} - a^*_{-i,t'}\right\| - \frac{\varepsilon_i}{2}\right)\left(a_i - a^*_{i,t'}\right)^2\right)$$

By definition of ε_i,[17]

$$\left(\forall t' \in \mathcal{T}^i \backslash \{t\}\right) : \left\| a_{-i} - a^*_{-i,t'} \right\| > \frac{\varepsilon_i}{2}$$

whereas

$$\left\| a_{-i} - a^*_{-i,t} \right\| \leqslant \frac{\varepsilon_i}{2}$$

which suffices to imply that

$$\min_{t' \in \mathcal{T}^i} \left(\left(\left\| a_{-i} - a^*_{-i,t'} \right\| - \frac{\varepsilon_i}{2} \right) \left(a_i - a^*_{i,t'} \right)^2 \right) = \left(\left\| a_{-i} - a^*_{-i,t} \right\| - \frac{\varepsilon_i}{2} \right) \left(a_i - a^*_{i,t} \right)^2$$

and hence that

$$Arg \max_{a_i \in \left\{ a^*_{i,t''} \right\}_{t'' \in \mathcal{T}^i}} \left(\min_{t' \in \mathcal{T}^i} \left(\left(\left\| a_{-i} - a^*_{-i,t'} \right\| - \frac{\varepsilon_i}{2} \right) \left(a_i - a^*_{i,t'} \right)^2 \right) \right) = \left\{ a^*_{i,t} \right\}$$

and that $\phi^i(a_{-i}) = a^*_{i,t}$.

Also, notice that $\forall t, t' \in \mathcal{T}^i$, $t \neq t'$,

$$\overline{B_{\frac{\varepsilon_i}{2}}\left(a^*_{-i,t}\right)} \cap \overline{B_{\frac{\varepsilon_i}{2}}\left(a^*_{-i,t'}\right)} = \varnothing$$

from where it follows that ϕ^i is continuous. Since this function is also bounded, then, by Tietze's extension theorem, there exists $\Phi^i : A^{-i} \longrightarrow A^i$, continuous, such that

$$\left(\forall a_{-i} \in C^i\right) : \Phi^i(a_{-i}) = \phi^i(a_{-i})$$

Define now $U^i : A^i \times A^{-i} \longrightarrow \mathbb{R}$ by

$$U^i(a_i, a_{-i}) = -\left(a_i - \Phi^i(a_{-i})\right)^2$$

Clearly, U^i is continuous.

I now have to show that, so defined, $\left(U^i\right)_{i \in \mathcal{I}}$ Nash-rationalizes the data set. It is immediate that $\forall a_{-i} \in A^{-i}$, the function $U^i(\cdot, a_{-i})$ is differentiable and strongly concave. In order to show that

$$(\forall t \in \mathcal{T}) : \left(a^*_{i,t}\right)_{i \in \mathcal{I}} \in N\left(\left(\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right], U^i_{\overline{a}_t}\right)_{i \in \mathcal{I}}\right)$$

[17] This follows by triangle inequality: if for some $t' \in T^i \backslash \{t\}$, $\left\| a_{-i} - a^*_{-i,t'} \right\| \leqslant \frac{\varepsilon_i}{2}$, then

$$\begin{aligned} \left\| a^*_{-i,t} - a^*_{-i,t'} \right\| &= \left\| a^*_{-i,t} - a_i + a_i - a^*_{-i,t'} \right\| \\ &\leqslant \left\| a^*_{-i,t} - a_i \right\| + \left\| a_i - a^*_{-i,t'} \right\| \\ &\leqslant \frac{\varepsilon_i}{2} + \frac{\varepsilon_i}{2} \\ &= \varepsilon_i \\ &< \overline{\varepsilon}_i \end{aligned}$$

a contradiction.

let $i \in \mathcal{I}$ and $t \in \mathcal{T}$ and define $\Gamma_{i,t} = \left\{ t' \in \mathcal{T} | \, a^*_{-i,t'} = a^*_{-i,t} \right\}$. As before, one of the following three mutually exclusive cases must hold:

Case 1: $(\exists t' \in \Gamma_{i,t}) : a^*_{i,t'} \in \left(\underline{a}_{i,t'}, \overline{a}_{i,t'} \right)$

Case 2: $\left((\forall t' \in \Gamma_{i,t}) : a^*_{i,t'} = \underline{a}_{i,t'} \vee a^*_{i,t'} = \overline{a}_{i,t'} \right) \wedge \left((\exists' \in \Gamma_{i,t}) : a^*_{i,t'} = \overline{a}_{i,t'} \right)$

Case 3: $(\forall t' \in \Gamma_{i,t}) : a^*_{i,t'} = \underline{a}_{i,t'}$

Suppose that case 1 holds. Let

$$t'_t = \min \left(Arg \min_{t' \in \Gamma_{i,t} : a^*_{i,t'} \in \left(\underline{a}_{i,t'}, \overline{a}_{i,t'} \right)} a^*_{i,t'} \right)$$

By construction, $t'_t \in \mathcal{T}^i$, $a^*_{-i,t'_t} = a^*_{-i,t}$ and, therefore

$$\begin{aligned} \Phi^i \left(a^*_{-i,t} \right) &= \Phi^i \left(a^*_{-i,t'_t} \right) \\ &= \phi^i \left(a^*_{-i,t'_t} \right) \\ &= a^*_{i,t'_t} \end{aligned}$$

from where, if $t'_t = t$,

$$\begin{aligned} a^*_{i,t} &= \arg \max_{a_i \in A^i} U^i \left(a_i, a^*_{-i,t} \right) \\ &= \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t} \right]} U^i \left(a_i, a^*_{-i,t} \right) \end{aligned}$$

Alternatively, suppose that $t \in \mathcal{T} \setminus \{t'_t\}$.

If $a^*_{i,t} \in \left(\underline{a}_{i,t}, \overline{a}_{i,t} \right)$, then by, condition (2), $a^*_{i,t'_t} < \overline{a}_{i,t}$ and $a^*_{i,t} < \overline{a}_{i,t'_t}$, whereas, by condition (3), $a^*_{i,t'_t} > \underline{a}_{i,t}$ and $a^*_{i,t} > \underline{a}_{i,t'_t}$. Then, by condition (1), $a^*_{i,t} = a^*_{i,t'_t}$, and, therefore,

$$\begin{aligned} a^*_{i,t} &= \arg \max_{a_i \in A^i} U^i \left(a_i, a^*_{-i,t} \right) \\ &= \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t} \right]} U^i \left(a_i, a^*_{-i,t} \right) \end{aligned}$$

On the other hand, suppose that $a^*_{i,t} = \overline{a}_{i,t}$. If $\overline{a}_{i,t} \leqslant a^*_{i,t'_t}$ it is obvious that

$$\begin{aligned} \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t} \right]} U^i \left(a_i, a^*_{-i,t} \right) &= \overline{a}_{i,t} \\ &= a^*_{i,t} \end{aligned}$$

whereas if $\overline{a}_{i,t} > a^*_{i,t'_t}$, there are four possibilities: (i) if $\underline{a}_{i,t} < a^*_{i,t'_t}$ and $\overline{a}_{i,t'_t} \leqslant \overline{a}_{i,t}$, then $a^*_{i,t} \in \left(\underline{a}_{i,t}, \overline{a}_{i,t} \right]$ and $\overline{a}_{i,t'_t} \leqslant a^*_{i,t}$, which implies, by condition (2), that $a^*_{i,t'_t} = \overline{a}_{i,t'_t}$ and contradicts the fact that $a^*_{i,t'_t} \in \left(\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t} \right)$; (ii) If $\underline{a}_{i,t} < a^*_{i,t'_t}$, $\overline{a}_{i,t'_t} > \overline{a}_{i,t}$ and $\underline{a}_{i,t'_t} \leqslant a^*_{i,t}$, then $a^*_{i,t'_t} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t} \right]$ and $a^*_{i,t} \in \left[\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t} \right]$ and, therefore, by condition (1), $a^*_{i,t} = a^*_{i,t'_t}$, which implies that

$$\begin{aligned} a^*_{i,t} &= \arg\max_{a_i \in A^i} U^i\left(a_i, a^*_{-i,t}\right) \\ &= \arg\max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right) \end{aligned}$$

(iii) If $\underline{a}_{i,t} < a^*_{i,t'_t}$, $\overline{a}_{i,t'_t} > \overline{a}_{i,t}$ and $\underline{a}_{i,t'_t} > a^*_{i,t}$, then

$$\begin{aligned} \overline{a}_{i,t} &= a^*_{i,t} \\ &< \underline{a}_{i,t'_t} \\ &< a^*_{i,t'_t} \\ &< \overline{a}_{i,t} \end{aligned}$$

which is an obvious contradiction; (iv) finally, if $\underline{a}_{i,t} \geqslant a^*_{i,t'_t}$, then $a^*_{i,t'_t} \in \left[\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t}\right)$ and $\underline{a}_{i,t} \geqslant a^*_{i,t'_t}$ imply by condition (3) that $a^*_{i,t} = \underline{a}_{i,t}$, which contradicts the fact that $a^*_{i,t} = \overline{a}_{i,t} > \underline{a}_{i,t}$.

Finally, suppose that $a^*_{i,t} = \underline{a}_{i,t}$. If $a^*_{i,t'_t} \leqslant \underline{a}_{i,t}$ it is obvious that

$$a^*_{i,t} = \arg\max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right)$$

whereas if $a^*_{i,t'_t} > \underline{a}_{i,t}$, there are three possibilities: (i) if $\overline{a}_{i,t} \leqslant a^*_{i,t'_t}$, then $a^*_{i,t'_t} \in \left(\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t}\right]$ and $\overline{a}_{i,t} \leqslant a^*_{i,t'_t}$ imply, by condition (2), that $a^*_{i,t} = \overline{a}_{i,t}$, which contradicts the fact that $a^*_{i,t} = \underline{a}_{i,t} < \overline{a}_{i,t}$, (ii) if $\overline{a}_{i,t} > a^*_{i,t'_t}$ and $\underline{a}_{i,t'_t} \geqslant a^*_{i,t}$, then $a^*_{i,t} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right)$ and $a^*_{i,t} \leqslant \underline{a}_{i,t'_t}$ imply, by condition (3), that $a^*_{i,t'_t} = \underline{a}_{i,t'_t}$, which contradicts the fact that $a^*_{i,t'_t} \in \left(\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t}\right)$; (iii) finally, if $\overline{a}_{i,t} > a^*_{i,t'_t}$ and $\underline{a}_{i,t'_t} < a^*_{i,t}$, then $a^*_{i,t'_t} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]$ and $a^*_{i,t} \in \left[\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t}\right]$ and, therefore, by condition (1), $a^*_{i,t} = a^*_{i,t'_t}$, which implies that

$$\begin{aligned} a^*_{i,t} &= \underline{a}_{i,t} \\ &< a^*_{i,t'} \\ &= a^*_{i,t} \end{aligned}$$

which is an obvious contradiction.

Suppose now that case 2 holds. Let

$$t'_t = \min\left(Arg \max_{t' \in \Gamma_{i,t} : a^*_{i,t'} = \overline{a}_{i,t'}} a^*_{i,t'}\right)$$

By construction, $t'_t \in \mathcal{T}^i$, $a^*_{-i,t'_t} = a^*_{-i,t}$ and, therefore,

$$\begin{aligned} \Phi^i\left(a^*_{-i,t'_t}\right) &= \Phi^i\left(a^*_{-i,t'_t}\right) \\ &= \phi^i\left(a^*_{-i,t'_t}\right) \\ &= a^*_{i,t'_t} \end{aligned}$$

from where, if $t_t' = t$,

$$\begin{aligned} a^*_{i,t,} &= \arg\max_{a_i \in A^i} U^i\left(a_i, a^*_{-i,t}\right) \\ &= \arg\max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right) \end{aligned}$$

Now, suppose that $t \in \mathcal{T} \setminus \{t_t'\}$.

If $\overline{a}_{i,t} \leqslant a^*_{i,t_t'}$, then, by condition (3), $a^*_{i,t} = \overline{a}_{i,t}$, whereas by construction

$$\overline{a}_{i,t} = \arg\max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right)$$

Alternatively, $\overline{a}_{i,t} > a^*_{i,t_t'}$, from where, by construction, $a^*_{i,t} = \underline{a}_{i,t}$. There are three possibilities: (i) If $\underline{a}_{i,t} \leqslant \underline{a}_{i,t_t'}$, then, by condition (3), $a^*_{i,t_t'} = \underline{a}_{i,t_t'} < \overline{a}_{i,t_t'}$, which is a contradiction, because $a^*_{i,t_t'} = \overline{a}_{i,t_t'}$; (ii) if $\underline{a}_{i,t_t'} < \underline{a}_{i,t} \leqslant a^*_{i,t_t'}$, then $\underline{a}_{i,t} \leqslant a^*_{i,t_t'} \leqslant \overline{a}_{i,t}$ and $\underline{a}_{i,t_t'} < \underline{a}_{i,t} = a^*_{i,t} \leqslant a^*_{i,t_t'} = \overline{a}_{i,t_t'}$, which implies, by condition (1), that $a^*_{i,t} = a^*_{i,t_t'}$ and hence that

$$\begin{aligned} a^*_{i,t} &= \arg\max_{a_i \in A^i} U^i\left(a_i, a^*_{-i,t}\right) \\ &= \arg\max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right) \end{aligned}$$

(iii) if $a^*_{i,t_t'} < \underline{a}_{i,t} = a^*_{i,t}$, then, by construction,

$$a^*_{i,t} = \arg\max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right)$$

Finally, suppose that case 3 holds. Let

$$t_t' = \min\left(Arg \min_{t' \in \Gamma_{i,t}} a^*_{i,t'}\right)$$

By construction, $t_t' \in \mathcal{T}^i$, $a^*_{-i,t_t'} = a^*_{-i,t}$ and, therefore

$$\begin{aligned} \Phi^i\left(a^*_{-i,t}\right) &= \Phi^i\left(a^*_{-i,t_t'}\right) \\ &= \phi^i\left(a^*_{-i,t_t'}\right) \\ &= a^*_{i,t_t'} \end{aligned}$$

Since $a^*_{i,t} = \underline{a}_{i,t} \geqslant a^*_{i,t_t'}$, it follows that

$$a^*_{i,t} = \arg\max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right)$$

This implies that

$$(\forall t \in \mathcal{T}) : \left(a^*_{i,t}\right)_{i \in \mathcal{I}} \in N\left(\left(\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right], U^i_{\overline{a}_t}\right)_{i \in \mathcal{I}}\right)$$

Let

$$\varepsilon = \min_{i \in \mathcal{I}} \varepsilon_i$$

Since $I < \infty$, it follows that $\varepsilon > 0$. Define for each $t \in \mathcal{T}$

$$\mathcal{O}\left(\left(a^*_{i,t}\right)_{i \in \mathcal{I}}\right) = B_{\frac{\varepsilon}{2}}\left(\left(a^*_{i,t}\right)_{i \in \mathcal{I}}\right)$$

Let $t \in \mathcal{T}$. All that remains to be shown now is that if

$$(a_i)_{i \in \mathcal{I}} \in \left(\mathcal{O}\left(\left(a^*_{i,t}\right)_{i \in \mathcal{I}}\right) \cap \prod_{i \in \mathcal{I}} A^i\right) \Big\backslash \left\{\left(a^*_{i,t}\right)_{i \in \mathcal{I}}\right\}$$

then

$$(a_i)_{i \in \mathcal{I}} \notin N\left(\left(\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right], U^i_{\overline{a}_t}\right)_{i \in \mathcal{I}}\right)$$

To see this, notice that if $(a_i)_{i \in \mathcal{I}} \neq \left(a^*_{i,t}\right)_{i \in \mathcal{I}}$, then for some $i \in \mathcal{I}$, $a_i \neq a^*_{i,t}$. Fix one such i. By construction,

$$\left\|(a_j)_{j \in \mathcal{I}} - \left(a^*_{j,t}\right)_{j \in \mathcal{I}}\right\| < \frac{\varepsilon}{2} \leqslant \frac{\varepsilon_i}{2}$$

so that

$$\left\|a_{-i} - a^*_{-i,t}\right\| < \frac{\varepsilon_i}{2}$$

Consider first the case $t \in \mathcal{T}^i$. Clearly,[18]

$$\left(\forall t' \in \mathcal{T}^i \backslash \{t\}\right) : \left\|a_{-i} - a^*_{-i,t'}\right\| > \frac{\varepsilon_i}{2}$$

which suffices to imply that

$$\min_{t' \in \mathcal{T}^i}\left(\left(\left\|a_{-i} - a^*_{-i,t'}\right\| - \frac{\varepsilon_i}{2}\right)\left(a_i - a^*_{i,t'}\right)^2\right) = \left(\left\|a_{-i} - a^*_{-i,t}\right\| - \frac{\varepsilon_i}{2}\right)\left(a_i - a^*_{i,t}\right)^2$$

and, since $a_{-i} \in \overline{B_{\frac{\varepsilon_i}{2}}\left(a^*_{-i,t}\right)}$, that $\Phi^i(a_{-i}) = \phi^i(a_{-i}) = a^*_{i,t}$ and hence that

$$\begin{aligned} U^i(a_i, a_{-i}) &= -\left(a_i - a^*_{i,t}\right)^2 \\ &< 0 \\ &= -\left(a^*_{i,t} - a^*_{i,t}\right)^2 \\ &= U^i\left(a^*_{i,t}, a_{-i}\right) \end{aligned}$$

so that

$$a_i \notin Arg \max_{\widehat{a}_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i(\widehat{a}_i, a_{-i})$$

and

$$(a_i)_{i \in \mathcal{I}} \notin N\left(\left(\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right], U^i_{\overline{a}_t}\right)_{i \in \mathcal{I}}\right)$$

[18] See footnote 17.

If, on the other hand, $t \in \mathcal{T} \backslash \mathcal{T}^i$, then, as before, if $\Gamma_{i,t} = \left\{ t' \in \mathcal{T} | \, a^*_{-i,t'} = a^*_{-i,t} \right\}$, one of the following cases holds:

Case 1: $(\exists t' \in \Gamma_{i,t}) : a^*_{i,t'} \in \left(\underline{a}_{i,t'}, \overline{a}_{i,t'} \right)$

Case 2: $\left((\forall t' \in \Gamma_{i,t}) : a^*_{i,t'} = \underline{a}_{i,t'} \vee a^*_{i,t'} = \overline{a}_{i,t'} \right) \wedge \left((\exists' \in \Gamma_{i,t}) : a^*_{i,t'} = \overline{a}_{i,t'} \right)$

Case 3: $(\forall t' \in \Gamma_{i,t}) : a^*_{i,t'} = \underline{a}_{i,t'}$

Suppose that case 1 holds. Let

$$t'_t = \min \left(Arg \min_{t' \in \Gamma_{i,t} : a^*_{i,t'} \in \left(\underline{a}_{i,t'}, \overline{a}_{i,t'} \right)} a^*_{i,t'} \right)$$

By construction, $t'_t \in \mathcal{T}$, $a^*_{-i,t'_t} = a^*_{-i,t}$ and, since $a_{-i} \in B_{\frac{\varepsilon_i}{2}} \left(a^*_{-i,t} \right)$

$$\begin{aligned} \Phi^i \left(a^*_{-i} \right) &= \Phi^i \left(a^*_{-i,t} \right) \\ &= \Phi^i \left(a^*_{-i,t'_t} \right) \\ &= \phi^i \left(a^*_{-i,t'_t} \right) \\ &= a^*_{i,t'_t} \end{aligned}$$

If $a^*_{i,t} \in \left(\underline{a}_{i,t}, \overline{a}_{i,t} \right)$, then by, condition (2), $a^*_{i,t'_t} < \overline{a}_{i,t}$ and $a^*_{i,t} < \overline{a}_{i,t'_t}$, whereas, by condition (3), $a^*_{i,t'_t} > \underline{a}_{i,t}$ and $a^*_{i,t} > \underline{a}_{i,t'_t}$. Then, by condition (1), $a^*_{i,t} = a^*_{i,t'_t}$, and, therefore,

$$\begin{aligned} a^*_{i,t} &= \arg \max_{a_i \in A^i} U^i \left(a_i, a^*_{-i,t} \right) \\ &= \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t} \right]} U^i \left(a_i, a^*_{-i,t} \right) \end{aligned}$$

On the other hand, suppose that $a^*_{i,t} = \overline{a}_{i,t}$. If $\overline{a}_{i,t} \leqslant a^*_{i,t'_t}$ it is obvious that

$$\begin{aligned} \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t} \right]} U^i \left(a_i, a^*_{-i,t} \right) &= \overline{a}_{i,t} \\ &= a^*_{i,t} \end{aligned}$$

whereas if $\overline{a}_{i,t} > a^*_{i,t'_t}$, there are four possibilities: (i) if $\underline{a}_{i,t} < a^*_{i,t'_t}$ and $\overline{a}_{i,t'_t} \leqslant \overline{a}_{i,t}$, then $a^*_{i,t} \in \left(\underline{a}_{i,t}, \overline{a}_{i,t} \right]$ and $\overline{a}_{i,t'_t} \leqslant a^*_{i,t}$, which implies, by condition (2), that $a^*_{i,t'_t} = \overline{a}_{i,t'_t}$ and contradicts the fact that $a^*_{i,t'_t} \in \left(\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t} \right)$; (ii) If $\underline{a}_{i,t} < a^*_{i,t'_t}$, $\overline{a}_{i,t'_t} > \overline{a}_{i,t}$ and $\underline{a}_{i,t'_t} \leqslant a^*_{i,t}$, then $a^*_{i,t'_t} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t} \right]$ and $a^*_{i,t} \in \left[\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t} \right]$ and, therefore, by condition (1), $a^*_{i,t} = a^*_{i,t'_t}$, which implies that

$$\begin{aligned} a^*_{i,t} &= \arg \max_{a_i \in A^i} U^i \left(a_i, a^*_{-i,t} \right) \\ &= \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t} \right]} U^i \left(a_i, a^*_{-i,t} \right) \end{aligned}$$

(iii) If $\underline{a}_{i,t} < a^*_{i,t'_t}$, $\overline{a}_{i,t'_t} > \overline{a}_{i,t}$ and $\underline{a}_{i,t'_t} > a^*_{i,t}$, then

$$\begin{aligned} \overline{a}_{i,t} &= a^*_{i,t} \\ &< \underline{a}_{i,t'_t} \\ &< a^*_{i,t'_t} \\ &< \overline{a}_{i,t} \end{aligned}$$

which is an obvious contradiction; (iv) finally, if $\underline{a}_{i,t} \geqslant a^*_{i,t'_t}$, then $a^*_{i,t'_t} \in \left[\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t}\right)$ and $\underline{a}_{i,t} \geqslant a^*_{i,t'_t}$ imply by condition (3) that $a^*_{i,t} = \underline{a}_{i,t}$, which contradicts the fact that $a^*_{i,t} = \overline{a}_{i,t} > \underline{a}_{i,t}$.

Finally, suppose that $a^*_{i,t} = \underline{a}_{i,t}$. If $a^*_{i,t'_t} \leqslant \underline{a}_{i,t}$ it is obvious that

$$a^*_{i,t} = \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right)$$

whereas if $a^*_{i,t'_t} > \underline{a}_{i,t}$, there are three possibilities: (i) if $\overline{a}_{i,t} \leqslant a^*_{i,t'_t}$, then $a^*_{i,t'_t} \in \left(\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t}\right]$ and $\overline{a}_{i,t} \leqslant a^*_{i,t'_t}$ imply, by condition (2), that $a^*_{i,t} = \overline{a}_{i,t}$, which contradicts the fact that $a^*_{i,t} = \underline{a}_{i,t} < \overline{a}_{i,t}$, (ii) if $\overline{a}_{i,t} > a^*_{i,t'_t}$ and $\underline{a}_{i,t'_t} \geqslant a^*_{i,t}$, then $a^*_{i,t} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right)$ and $a^*_{i,t} \leqslant \underline{a}_{i,t'_t}$ imply, by condition (3), that $a^*_{i,t'_t} = \underline{a}_{i,t'_t}$, which contradicts the fact that $a^*_{i,t'_t} \in \left(\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t}\right)$; (iii) finally, if $\overline{a}_{i,t} > a^*_{i,t'_t}$ and $\underline{a}_{i,t'_t} < a^*_{i,t}$, then $a^*_{i,t'_t} \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]$ and $a^*_{i,t} \in \left[\underline{a}_{i,t'_t}, \overline{a}_{i,t'_t}\right]$ and, therefore, by condition (1), $a^*_{i,t} = a^*_{i,t'_t}$, which implies that

$$\begin{aligned} a^*_{i,t} &= \underline{a}_{i,t} \\ &< a^*_{i,t'} \\ &= a^*_{i,t} \end{aligned}$$

which is an obvious contradiction.

At any rate, it follows that

$$a_i \notin Arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i\left(a_i, a^*_{-i,t}\right)$$

Suppose now that case 2 holds. Let

$$t'_t = \min\left(Arg \max_{t' \in \Gamma_{i,t} : a^*_{i,t'} = \overline{a}_{i,t'}} a^*_{i,t'}\right)$$

By construction, $t'_t \in \mathcal{T}$, $a^*_{-i,t'_t} = a^*_{-i,t}$ and, since $a_{-i} \in B_{\frac{\varepsilon_i}{2}}\left(a^*_{-i,t}\right)$

$$\begin{aligned} \Phi^i\left(a_{-i}\right) &= \Phi^i\left(a^*_{-i,t}\right) \\ &= \Phi^i\left(a^*_{-i,t'_t}\right) \\ &= \phi^i\left(a^*_{-i,t'_t}\right) \\ &= a^*_{i,t'_t} \end{aligned}$$

If $\overline{a}_{i,t} \leqslant a^*_{i,t'_t}$, then, by condition (3), $a^*_{i,t} = \overline{a}_{i,t}$, whereas by construction

$$\overline{a}_{i,t} = \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i \left(a_i, a^*_{-i,t}\right)$$

Alternatively, $\overline{a}_{i,t} > a^*_{i,t'_t}$, from where, by construction, $a^*_{i,t} = \underline{a}_{i,t}$. There are three possibilities: (i) If $\underline{a}_{i,t} \leqslant \underline{a}_{i,t'_t}$, then, by condition (3), $a^*_{i,t'_t} = \underline{a}_{i,t'_t} < \overline{a}_{i,t'_t}$, which is a contradiction; (ii) if $\underline{a}_{i,t'_t} < \underline{a}_{i,t} \leqslant a^*_{i,t'_t}$, then $\underline{a}_{i,t} \leqslant a^*_{i,t'_t} \leqslant \overline{a}_{i,t}$ and $\underline{a}_{i,t'_t} < \underline{a}_{i,t} = a^*_{i,t} \leqslant a^*_{i,t'_t} = \overline{a}_{i,t'_t}$, which implies, by condition (1), that $a^*_{i,t} = a^*_{i,t'_t}$ and hence that

$$\begin{aligned} a^*_{i,t} &= \arg \max_{a_i \in A^i} U^i \left(a_i, a^*_{-i,t}\right) \\ &= \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i \left(a_i, a^*_{-i,t}\right) \end{aligned}$$

(iii) if $a^*_{i,t'_t} < \underline{a}_{i,t} = a^*_{i,t}$, then, by construction,

$$a^*_{i,t} = \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i \left(a_i, a^*_{-i,t}\right)$$

At any rate,

$$a_{-i} \notin Arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i \left(a_i, a^*_{-i,t}\right)$$

Finally, suppose that case 3 holds. Let

$$t'_t = \min \left(Arg \min_{t' \in \Gamma_{i,t}} a^*_{i,t'} \right)$$

By construction, $t'_t \in \mathcal{T}$, $a^*_{-i,t'_t} = a^*_{-i,t}$ and, since $a_{-i} \in B_{\frac{\varepsilon_i}{2}} \left(a^*_{-i,t}\right)$

$$\begin{aligned} \Phi^i \left(a^*_{-i}\right) &= \Phi^i \left(a^*_{-i,t}\right) \\ &= \Phi^i \left(a^*_{-i,t'_t}\right) \\ &= \phi^i \left(a^*_{-i,t'_t}\right) \\ &= a^*_{i,t'_t} \end{aligned}$$

Since $a^*_{i,t} = \underline{a}_{i,t} \geqslant a^*_{i,t'_t}$, it follows that

$$a^*_{i,t} = \arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i \left(a_i, a^*_{-i,t}\right)$$

and, therefore

$$a_{-i} \notin Arg \max_{a_i \in \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]} U^i \left(a_i, a^*_{-i,t}\right)$$

■

It follows directly from theorem 1 that the conditions of theorem 2 are also necessary for this stronger version of rationalizability that requires local uniqueness. What theorem 2 implies, then, is that, from an empirical perspective, these two versions of the Nash-behavior hypothesis are indistinguishable. Put another way, in the context assumed here, the hypothesis of determinacy of Nash equilibria is unfalsifiable: given a data set that is Nash-rationalizable, one can never rule out the possibility that the observed equilibria are locally unique.

3.3 Harshness of the restrictions:

Following the Popperian method, the derivation of the testable implications of a theory ought to be followed by an assessment of how harsh these implications are. A harsh test is one that the researcher, before observing the data, would expect the theory to fail. Simple observation of the conditions of theorem 1 reveals that they are extremely mild. The three of them apply, individual-wise and for pairs of observations, conditionally on all the other players keeping their actions unchanged. Now, suppose that there are two players who choose their actions a_i^* randomly, uniformly over $\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]$, which is never a singleton set. In this case, the probability that for some $i \in \mathcal{I}$, there exist $t, t' \in \mathcal{T}$ such that $a_{-i,t}^* = a_{-i,t'}^*$ is zero. With all likelihood, one should expect the hypothesis of Nash behavior to pass the test of theorem 1, notwithstanding the fact that it is false.

Formally, this comes from the following straightforward corollary of theorem 1:

Corollary 1 *Given a data set*

$$\left(\left(a_{i,t}^*, \underline{a}_{i,t}, \overline{a}_{i,t}\right)_{i \in \mathcal{I}}\right)_{t=1}^{T}$$

if for each $t, t' \in \mathcal{T}$ *such that* $t \neq t'$, *there exist* $i', i'' \in \mathcal{I}$ *such that*

$$\begin{aligned} i' &\neq i'' \\ a_{i',t}^* &\neq a_{i',t'}^* \\ a_{i'',t}^* &\neq a_{i'',t'}^* \end{aligned}$$

then, the data set is Nash-rationalizable.

Proof. Fix $i \in \mathcal{I}$ and $t, t' \in \mathcal{T}$. Since $\exists i', i'' \in \mathcal{I}$ such that

$$\begin{aligned} i' &\neq i'' \\ a_{i',t}^* &\neq a_{i',t'}^* \\ a_{i'',t}^* &\neq a_{i'',t'}^* \end{aligned}$$

then $a_{-i,t}^* \neq a_{-i,t'}^*$. Since this is true $\forall i \in \mathcal{I}$ and $\forall t, t' \in \mathcal{T}$, then, by theorem 1, the data set is Nash-rationalizable. ■

In an informal sense, the implication of this corollary is that the restrictions of theorem 1 have "zero measure" and hence, even before observing any data, researchers should expect the test to be passed. Given this result, the sufficiency of the conditions of theorem 1 becomes crucial: even an exhaustive list of empirical restrictions of Nash behavior fails to constitute a harsh test of its principles. Moreover, it follows from theorem 2, that requiring local uniqueness of equilibria will not change this conclusion.

It is in this sense that I sustained in section 2 that the criticism of the Nash solution for its extreme informational assumptions appears less severe in this context. Even under the strong presumption that each player foresees the actions of his opponents with perfect accuracy, one will almost always fail to reject the hypothesis of Nash behavior. From an empirical point of view, one could almost never reject such a hypothesis of perfect accuracy. Hence, only upon rejection of the hypothesis would one need to explore explanations of the observed behavior based on beliefs, and not only on preferences.

4 Cooperative behavior:

Another point of the Popperian postulate is the comparison of theories with distinguished alternative theories, in order to asses their relative predictive abilities. The most prominent alternative behavioral assumption is that of cooperative behavior by agents, which in its most basic form would assume that individuals cooperate so as to obtain Pareto-efficient outcomes. I now study the testable implications of the Pareto solution in the context developed in the previous section.

As in noncooperative behavior, I want to study conditions under which the observed data set may be the result of cooperative behavior, in which for each observation $t \in \mathcal{T}$ agents jointly choose a Pareto-efficient outcome, $\left(a_{i,t}^*\right)_{i\in\mathcal{I}}$, according to their preferences, from the collectively feasible domain $\prod_{i\in\mathcal{I}}\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right]$. I again propose a weak definition of rationalizability, which requires that each observed outcome be Pareto-efficient, but does not impose that it be the only Pareto-efficient outcome, nor does it presume that all the Pareto sets of the observed feasible sets have been captured by the finite data set. The risk of trivial results arises again: if individuals are indifferent between all outcomes of the games, or if one can assume that there exists at least two players who are global antagonist, in the sense that for any domain an improvement for one of them means that the other one is worse off, then any data set is rationalizable. Once again, these possibilities are excluded via a concavity assumption that restricts the class of preferences allowed in the rationalizations.

Before the actual definition of Pareto-rationalizability, the following notation needs to be introduced: given a set $\mathcal{I}$ of players, each of whom has a set B^i of feasible actions and preferences represented by

$$V^i : \prod_{j\in\mathcal{I}} B_j \longrightarrow \mathbb{R}$$

let $P\left(\left(B^i, V^i\right)_{i\in\mathcal{I}}\right)$ represent the set of Pareto-efficient outcomes of the game $\left(B^i, V^i\right)_{i\in\mathcal{I}}$.

With this, the data sets that are considered consistent with cooperative behavior are defined as follows:

Definition 3 *A data set*

$$\left(\left(a_{i,t}, \underline{a}_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)_{t=1}^{T}$$

is (nontrivially, weakly) Pareto-rationalizable if for each $i \in \mathcal{I}$ there exists $U^i : \prod_{j\in\mathcal{I}} A_j \longrightarrow \mathbb{R}$, differentiable and strongly concave, such that

$$(\forall t \in \mathcal{T}) : (a_{i,t})_{i\in\mathcal{I}} \in P\left(\left(\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right], U^i_{\underline{a}_t,\overline{a}_t}\right)_{i\in\mathcal{I}}\right)$$

In this case, it is said that $\left(U^i\right)_{i\in\mathcal{I}}$ Pareto-rationalizes the data.

For notational simplicity, throughout this section I do not distinguish the actions of each agent from those of his opponents. The obvious reason is that in the case of Pareto-efficient decisions, all variables are treated equivalently by the players (the ordinal effect of a_{-i} on U^i does matter). What is conceptually important is that for this same reason I am restricting the class of utility functions allowed for rationalization to a strict subset of the class allowed for noncooperative behavior, which only required strong concavity with respect to own actions. In that sense, I am using a stronger criterion of rationalizability for the hypothesis of cooperation. However, as the following results show, not even under this stronger criterion is cooperative behavior falsifiable.

In the following result, I ignore conjunctional constraints, and study the Pareto set over the whole collective domain.

Theorem 3 *For any finite subset $\{a_t\}_{t=1}^{T}$ of $\prod_{i\in\mathcal{I}} A^i$, there exists $\left(U^i\right)_{i\in\mathcal{I}}$ such that for each $i \in \mathcal{I}$, $U^i : \prod_{j\in\mathcal{I}} A_j \longrightarrow \mathbb{R}$ is differentiable and strongly concave, and*

$$\{a_t\}_{t=1}^{T} \subseteq P\left(\left(A^i, U^i\right)_{i\in\mathcal{I}}\right)$$

Proof. Assume, without loss of generality, that for each $i \in \mathcal{I}$, $\underline{a}_i = 0$ (and $\overline{a}_i > 0$). It follows from lemma 1 in appendix 7 that there exists $v \in \mathbb{R}^I_{++}$ such that for each $t, t' \in \mathcal{T}$, $t \neq t'$, it is true that $v \cdot a_t \neq v \cdot a_{t'}$. Fix one such $v \in \mathbb{R}^I$ and define, for each $t \in \mathcal{T}$, $V_t = v \cdot a_t$. Clearly,

$$(\forall t, t' \in \mathcal{T}) : t \neq t' \Longrightarrow V_t \neq V_{t'}$$

Consider the sequence $(a_t, v)_{t=1}^{T}$. I now show that this sequence satisfies condition 3 (see appendix 8.) Let $N \leqslant T$ and $\{\tau_1, ..., \tau_N\} \subseteq \mathcal{T}$. Suppose that $\forall n \in \{1, ..., N-1\}$

$$v \cdot \left(a_{\tau_{n+1}} - a_{\tau_n}\right) \leqslant 0$$

Then, by transitivity

$$V_{\tau_N} \leqslant V_{\tau_1}$$

and, since $\forall t, t' \in \mathcal{T} : t \neq t' \Longrightarrow V_t \neq V_{t'}$, we have that if $a_{\tau_N} \neq a_{\tau_1}$, then

$$V_{\tau_N} < V_{\tau_1}$$

which implies that

$$v \cdot (a_{\tau_N} - a_{\tau_1}) < 0$$

which implies that condition 2 is satisfied. In this case, condition 3 is immediate.

Since $v \in \mathbb{R}^I_{++}$, by theorem 6 in appendix 8, there exists a function $V^+ : \prod_{i \in \mathcal{I}} A^i \longrightarrow \mathbb{R}$, continuously differentiable and strongly concave, such that

$$(\forall t \in \mathcal{T}) \left(\forall a \in \prod_{i \in \mathcal{I}} A^i \right) : V^+(a) > V^+(a_t) \Longrightarrow v \cdot a > v \cdot a_t$$

and,

$$(\forall t \in \mathcal{T}) \left(\forall a \in \prod_{i \in \mathcal{I}} A^i \right) : v \cdot a < v \cdot a_t \Longrightarrow V^+(a) < V^+(a_t)$$

On the other hand, consider the sequence $\left(\overline{\overline{a}} - a_t, v\right)_{t=1}^T$, where $\overline{\overline{a}} = \left(\overline{\overline{a}}_i\right)_{i \in \mathcal{I}}$. I now show that this sequence also satisfies condition 3. As before, let $N \leqslant T$ and $\{\tau_1, ..., \tau_N\} \subseteq \mathcal{T}$. Suppose that $\forall n \in \{1, ..., N-1\}$

$$v \cdot \left(\left(\overline{\overline{a}} - a_{\tau_{n+1}}\right) - \left(\overline{\overline{a}} - a_{\tau_n}\right) \right) \leqslant 0$$

Then, $\forall n \in \{1, ..., N-1\}$

$$v \cdot \left(a_{\tau_{n+1}} - a_{\tau_n}\right) \geqslant 0$$

and, by transitivity,

$$V_{\tau_N} \geqslant V_{\tau_1}$$

Hence, since $\forall t, t' \in \mathcal{T} : t \neq t' \Longrightarrow V_t \neq V_{t'}$, we have that if $a_{\tau_N} \neq a_{\tau_1}$, then

$$V_{\tau_N} > V_{\tau_1}$$

which implies that

$$v \cdot (a_{\tau_N} - a_{\tau_1}) > 0$$

and that

$$v \cdot \left(\left(\overline{\overline{a}} - a_{\tau_N}\right) - \left(\overline{\overline{a}} - a_{\tau_1}\right) \right) < 0$$

which implies that condition 2 is satisfied. Again, condition 3 is here immediate.

Since $v \in \mathbb{R}^I_{++}$, it follows once again from theorem 6, that there exists a function $V^- : \prod_{i \in \mathcal{I}} A^i \longrightarrow \mathbb{R}$, differentiable and strongly concave, such that

$$(\forall t \in \mathcal{T}) \left(\forall a \in \prod_{i \in \mathcal{I}} A^i \right) : V^-(a) > V^-\left(\overline{\overline{a}} - a_t\right) \Longrightarrow v \cdot a > v \cdot \left(\overline{\overline{a}} - a_t\right)$$

and

$$(\forall t \in \mathcal{T}) \left(\forall a \in \prod_{i \in \mathcal{I}} A^i \right) : v \cdot a < v \cdot \left(\overline{\overline{a}} - a_t \right) \Longrightarrow V^- (a) < V^- \left(\overline{\overline{a}} - a_t \right)$$

Now, since $I \geqslant 2$, it follows that $\mathcal{I} \backslash \{I\} \neq \varnothing$ and, then, for each $i \in \mathcal{I} \backslash \{I\}$, one can define $U^i : \prod_{i \in \mathcal{I}} A^i \longrightarrow \mathbb{R}$, as $U^i (a) = V^+ (a)$, and define $U^I : \prod_{i \in \mathcal{I}} A^i \longrightarrow \mathbb{R}$, as $U^I (a) = V^- \left(\overline{\overline{a}} - a \right)$. That $\forall i \in \mathcal{I} \{I\}$, U^i is differentiable and strongly concave is obvious, whereas U^I is differentiable since so are V^- and the mapping $a \longmapsto \overline{\overline{a}} - a$, and is strongly concave since so is V^- while $a \longmapsto \overline{\overline{a}} - a$ is affine.

I finally show that

$$\{a_t\}_{t=1}^T \subseteq P \left(\left(A^i, U^i \right)_{i \in \mathcal{I}} \right)$$

Fix $t \in \mathcal{T}$. Suppose that $a \in \prod_{i \in \mathcal{I}} A^i$ is such that for $i \in \mathcal{I} \backslash \{I\}$ we have $U^i (a) > U^i (a_t)$. Then $V^+ (a) > V^+ (a_t)$, so that $v \cdot a > v \cdot a_t$ and, therefore, $v \cdot \left(\overline{\overline{a}} - a \right) < v \cdot \left(\overline{\overline{a}} - a_t \right)$, so that $V^- \left(\overline{\overline{a}} - a \right) < V^- \left(\overline{\overline{a}} - a_t \right)$ and $U^I (a) < U^I (a_t)$. If, on the other hand $a \in \prod_{i \in \mathcal{I}} A^i$ satisfies that $U^I (a) > U^I (a_t)$, then $V^- \left(\overline{\overline{a}} - a \right) > V^- \left(\overline{\overline{a}} - a_t \right)$, so that $v \cdot \left(\overline{\overline{a}} - a \right) > v \cdot \left(\overline{\overline{a}} - a_t \right)$ and, therefore, $v \cdot a < v \cdot a_t$, from where $V^+ (a) < V^+ (a_t)$, implying that for all $i \in \mathcal{I} \backslash \{I\}$, $U^i (a) < U^i (a_t)$. ■

What the previous proof does is to make player I antagonist to all the rest of the players at each a_t. This does not trivialize the result, because the antagonism does not occur globally, but only locally: global antagonists could not all have strongly concave utility functions.[19] The theorem applies more generally than the present context, as it holds for any number of players (greater than or equal to two) and any number of dimensions in the collective domain (that is, it does not require that the number of players and the number of dimensions be the same).

In the case dealt with here, where there may be constraints to choice, the implication is that the hypothesis of Pareto behavior is unfalsifiable, as implied by the following corollary:

Corollary 2 *Any data set*

$$\left(\left(a_{i,t}, \underline{a}_{i,t}, \overline{a}_{i,t} \right)_{i \in \mathcal{I}} \right)_{t=1}^T$$

is Pareto-rationalizable.

Proof. By theorem 3, for each $i \in \mathcal{I}$ there exists, $U^i : \prod_{j \in \mathcal{I}} A_j \longrightarrow \mathbb{R}$ differentiable and strongly concave, such that

$$\bigcup_{t=1}^T \{a_t\} \subseteq P \left(\left(A^i, U^i \right)_{i \in \mathcal{I}} \right)$$

[19] Of course, I could not have made the argument if I were also requiring monotonicity.

Fix $t \in \mathcal{T}$. Then,

$$\begin{aligned} a_t &\in P\left(\left(A^i, U^i\right)_{i \in \mathcal{I}}\right) \\ &\subseteq P\left(\left(\left[\underline{a}_{i,t}, \overline{a}_{i,t}\right], U^i_{\underline{a}_t, \overline{a}_t}\right)_{i \in \mathcal{I}}\right) \end{aligned}$$

because

$$\begin{aligned} a_t &\in \prod_{i \in \mathcal{I}} \left[\underline{a}_{i,t}, \overline{a}_{i,t}\right] \\ &\subseteq \prod_{i \in \mathcal{I}} A^i \end{aligned}$$

■

This result is interesting by itself, since it implies the unfalsifiability of a noticeable behavioral hypothesis in economics and in game theory.[20] Regarding the main focus of this paper, its implication is that the empirical restrictions of Nash behavior derived in section 3 are weak in yet another sense: not only are they extremely mild, but, also, they are weak in that whenever a data set passes the test, there is no way to rule out the possibility that it is coming from a totally different behavior by agents, namely that they are cooperating to obtain Pareto-efficient outcomes.

5 Further and stronger restrictions under non-cooperative behavior:

In view of the results of section 3, regarding the lack of predictive power of the Nash hypothesis, I now study whether further assumptions yield further or stronger testable restrictions.

Specifically, a reasonable criticism to the results of section 3 is that my definition of Nash-rationalizability imposes little structure regarding how a_{-i} enters U^i (only continuity) and that, therefore, in many specific cases such definition is inappropriate in the sense that the class of utility functions that it allows is too large. The reason why such a large class of preferences was used in section 3 is that it covers a large class of games. If one has theoretical reasons to impose other assumptions that reduce the class of preferences to be allowed, the question then arises of whether the restrictions of theorem 1 (or 2), which will of course remain necessary under subclasses of the original class, are all the restrictions of the theory. I now explore possible cases when, indeed, further and stronger testable implications exist.

[20] The result stands in contrast with the one in Chiappori (1988). As I have pointed out, however, Chiappori requires monotonicity in the utility functions, and has feasible sets in which there is trade-off between the player's actions. Because the context here does not have the second property, and because in its absence imposing monotonicity would trivialize the Nash-rationalizability problem, I have chosen not to impose any monotonicity requirement.

For the sake of simplicity, I henceforth assume that the following condition holds:

Condition 1 *For each $i \in \mathcal{I}$ and each $t \in \mathcal{T}$, $\underline{a}_i = \underline{a}_{i,t} = 0$.*

Under this assumption, one can redefine a data set as:

Definition 4 *A data set is a finite sequence*

$$\left(\left(a^*_{i,t}, \overline{a}_{i,t}\right)_{i \in \mathcal{I}}\right)_{t=1}^{T}$$

such that:

$$(\forall t \in \mathcal{T})\,(\forall i \in \mathcal{I}) : \left(0 < \overline{a}_{i,t} \leqslant \overline{a}_i \wedge a^*_{i,t} \in [0, \overline{a}_{i,t}]\right)$$

And then, the following corollary to theorem 1 is straightforward:

Corollary 3 *A data set*

$$\left(\left(a^*_{i,t}, \overline{a}_{i,t}\right)_{i \in \mathcal{I}}\right)_{t=1}^{T}$$

is Nash-rationalizable if, and only if, for all $t, t' \in \mathcal{T}$ and all $i \in \mathcal{I}$:

1.

$$\left.\begin{array}{c} a^*_{-i,t} = a^*_{-i,t'} \\ a^*_{i,t} \in [0, \overline{a}_{i,t'}] \\ a^*_{i,t'} \in [0, \overline{a}_{i,t}] \end{array}\right\} \implies a^*_{i,t} = a^*_{i,t'}$$

2.

$$\left.\begin{array}{c} a^*_{-i,t} = a^*_{-i,t'} \\ a^*_{i,t} \notin [0, \overline{a}_{i,t'}] \end{array}\right\} \implies a^*_{i,t'} = \overline{a}_{i,t'}$$

Proof. It suffices to show that conditions (1) and (2) here are equivalent to the restrictions of theorem 1, which, under condition 1, become: $\forall t, t' \in \mathcal{T}$ and $\forall i \in \mathcal{I}$:

(i):

$$\left.\begin{array}{c} a^*_{-i,t} = a^*_{-i,t'} \\ a^*_{i,t} \in [0, \overline{a}_{i,t'}] \\ a^*_{i,t'} \in [0, \overline{a}_{i,t}] \end{array}\right\} \implies a^*_{i,t} = a^*_{i,t'}$$

(ii):

$$\left.\begin{array}{c} a^*_{-i,t} = a^*_{-i,t'} \\ a^*_{i,t} \in (0, \overline{a}_{i,t}] \\ a^*_{i,t} \geqslant \overline{a}_{i,t'} \end{array}\right\} \implies a^*_{i,t'} = \overline{a}_{i,t'}$$

(iii):

$$\left.\begin{array}{c} a^*_{-i,t} = a^*_{-i,t'} \\ a^*_{i,t} = 0 \end{array}\right\} \implies a^*_{i,t'} = 0$$

Hence, suppose first that condition (1) and (2) here are satisfied. Condition (i) is immediate from 1.

For condition (ii), suppose that for some $t, t' \in \mathcal{T}$ and some $i \in \mathcal{I}$: $a^*_{-i,t} = a^*_{-i,t'}$, $a^*_{i,t} \in (0, \overline{a}_{i,t}]$ and $a^*_{i,t} \geqslant \overline{a}_{i,t'}$. If $a^*_{i,t} > \overline{a}_{i,t'}$, then, by condition (2), $a^*_{i,t'} = \overline{a}_{i,t'}$. Else, $a^*_{i,t} = \overline{a}_{i,t'}$, which implies that $a^*_{i,t} \in [0, \overline{a}_{i,t'}]$. If $a^*_{i,t'} > \overline{a}_{i,t}$, then

$$\begin{aligned} \overline{a}_{i,t'} &= a^*_{i,t} \\ &\leqslant \overline{a}_{i,t} \\ &< a^*_{i,t'} \\ &\leqslant \overline{a}_{i,t'} \end{aligned}$$

an obvious contradiction. Hence, it must be that $a^*_{i,t'} \leqslant \overline{a}_{i,t}$, in which case $a^*_{i,t'} \in [0, \overline{a}_{i,t}]$ and, therefore, by condition (1),

$$\begin{aligned} a^*_{i,t'} &= a^*_{i,t} \\ &= \overline{a}_{i,t'} \end{aligned}$$

For condition (iii), suppose that for some $t, t' \in \mathcal{T}$ and some $i \in \mathcal{I}$: $a^*_{-i,t} = a^*_{-i,t'}$ and $a^*_{i,t} = 0$. If $a^*_{i,t'} \in [0, \overline{a}_{i,t}]$, then, by condition (1), $a^*_{i,t'} = 0$. Else, $a^*_{i,t'} > \overline{a}_{i,t}$, which implies, by condition (2), that $a^*_{i,t} = \overline{a}_{i,t} > 0$, which is a contradiction.

Now, suppose that conditions (i), (ii) and (iii) are satisfied. Condition (1) is immediate from (i).

For condition (2), suppose that for some $t, t' \in \mathcal{T}$ and some $i \in \mathcal{I}$, $a^*_{-i,t} = a^*_{-i,t'}$, $a^*_{i,t} \notin [0, \overline{a}_{i,t'}]$. Then, $a^*_{i,t} > \overline{a}_{i,t'} > 0$, which implies that $a^*_{i,t} \in (0, \overline{a}_{i,t}]$ and $a^*_{i,t} \geqslant \overline{a}_{i,t'}$, and, therefore, by condition (ii) $a^*_{i,t'} = \overline{a}_{i,t'}$. ■

Again, condition (1) is a restatement of the Weak Axiom of Revealed Preferences,[21] which must hold for each individual, conditional on the other players keeping their actions and, therefore, the preferences of the particular individual unchanged. Condition (2) is now an axiom of revealed increasing monotonicity.

5.1 Further testable restrictions:

As a first direction to reduce the class of preferences allowed by the definitions of Nash-rationalizability, one may want to restrict the specific way in which actions by others affect individual best responses. Specifically, for the purposes of this subsection, suppose that there are only two players, so that $\mathcal{I} = \{1, 2\}$. Under some preferences for player $i \in \mathcal{I}$, actions by his opponent, j, are complementary (to his own) if whenever j increases the value of his choice, the marginal utility that i derives from his own action increases. If, on the contrary, this marginal utility decreases, then the actions of player j are substitutes for those of i, from the point of view of i.

Formally, suppose that for player $i \in \mathcal{I}$, $U^i : A^i \times A^{-i} \longrightarrow \mathbb{R}$ satisfies that for each $a_{-i} \in A^{-i}$, $U(\cdot, a_{-i})$ is differentiable. Then,

[21] Which in this case suffices for the Strong Axiom of Revealed Preferences.

Definition 5 *For $i \in \mathcal{I}$, a utility function $U^i : A^i \times A^{-i} \longrightarrow \mathbb{R}$ is said to exhibit strategic complementarity on $S \subseteq A^{-i}$, where S is an interval, if for all $a_{-i}, a'_{-i} \in S$,*

$$a_{-i} \leqslant a'_{-i} \Longrightarrow \left(\forall a_i \in A^i\right) : \frac{\partial U^i}{\partial a_i}\left(a_i, a_{-i}\right) \leqslant \frac{\partial U^i}{\partial a_i}\left(a_i, a'_{-i}\right)$$

Definition 6 *For $i \in \mathcal{I}$, a utility function $U^i : A^i \times A^{-i} \longrightarrow \mathbb{R}$ is said to exhibit strategic substitutability on $S \subseteq A^{-i}$, where S is an interval, if for all $a_{-i}, a'_{-i} \in S$,*

$$a_{-i} \leqslant a'_{-i} \Longrightarrow \left(\forall a_i \in A^i\right) : \frac{\partial U^i}{\partial a_i}\left(a_i, a_{-i}\right) \geqslant \frac{\partial U^i}{\partial a_i}\left(a_i, a'_{-i}\right)$$

Suppose also that one has *a priori* theoretical knowledge to imply, for example, that for each player the actions of his opponent are complementary on $[0, \widetilde{a}_{-i}] \subseteq \left[0, \overline{\overline{a}}_{-i}\right]$, and substitute on $\left[\widetilde{a}_{-i}, \overline{\overline{a}}_{-i}\right]$. Then, besides the restrictions of corollary 3, these two new hypotheses can be tested, and all the restrictions of Nash behavior in this new context are implied by the following result:

Theorem 4 *For each $i \in \mathcal{I}$, let $\widetilde{a}_i \in A^i$ be given. Let the data set*

$$\left(\left(a^*_{i,t}, \overline{a}_{i,t}\right)_{i \in \mathcal{I}}\right)_{t=1}^{T}$$

be given. There exists $\left(U^i\right)_{i \in \mathcal{I}}$ that Nash-rationalizes the data set, such that for each $i \in \mathcal{I}$, U^i exhibits strategic complementarity on $[0, \widetilde{a}_{-i}]$ and strategic substitutability on $\left[\widetilde{a}_{-i}, \overline{\overline{a}}_{-i}\right]$ if, and only if, for each $t, t' \in \mathcal{T}$ and each $i \in \mathcal{I}$:

1.
$$\left.\begin{array}{l} a^*_{-i,t} = a^*_{-i,t'} \\ a^*_{i,t} \in [0, \overline{a}_{i,t'}] \\ a^*_{i,t'} \in [0, \overline{a}_{i,t}] \end{array}\right\} \Longrightarrow a^*_{i,t} = a^*_{i,t'}$$

2.
$$\left.\begin{array}{l} a^*_{-i,t} = a^*_{-i,t'} \\ a^*_{i,t} \notin [0, \overline{a}_{i,t'}] \end{array}\right\} \Longrightarrow a^*_{i,t'} = \overline{a}_{i,t'}$$

3.
$$\left.\begin{array}{c} a^*_{-i,t} \in [0, \widetilde{a}_{-i}] \\ a^*_{-i,t'} \leqslant a^*_{-i,t} \end{array}\right\} \Longrightarrow a^*_{i,t} \in \left[\min\left\{\overline{a}_{i,t}, a^*_{i,t'}\right\}, \overline{a}_{i,t}\right]$$

4.
$$\left.\begin{array}{c} a^*_{-i,t} \in \left[\widetilde{a}_{-i}, \overline{\overline{a}}_{-i}\right] \\ a^*_{-i,t'} \geqslant a^*_{-i,t} \end{array}\right\} \Longrightarrow a^*_{i,t} \in \left[\min\left\{\overline{a}_{i,t}, a^*_{i,t'}\right\}, \overline{a}_{i,t}\right]$$

Proof. **Necessity:** Fix $i \in \mathcal{I}$. That conditions (1) and (2) are necessary follows from corollary 3. For condition (3), suppose that for some $t, t' \in \mathcal{T}$ we have that $a^*_{-i,t} \in [0, \widetilde{a}_{-i}]$ and $a^*_{i,t'} \leqslant a^*_{-i,t}$, yet $a^*_{i,t} < \min\left\{\overline{a}_{i,t}, a^*_{i,t'}\right\}$. By Nash rationalizability,

$$a^*_{i,t} \in Arg \max_{a_i \in [0, \overline{a}_{i,t}]} U^i\left(a_i, a^*_{-i,t}\right)$$

and then, by the Kühn-Tucker theorem, since $a^*_{i,t} < \overline{a}_{i,t}$, it must be true that

$$\frac{\partial U^i}{\partial a_i}\left(a^*_{i,t}, a^*_{-i,t}\right) \leqslant 0$$

Now, since U^i exhibits strategic complementarity on $[0, \widetilde{a}_{-i}]$ and $a^*_{-i,t'} \leqslant a^*_{-i,t} \leqslant \widetilde{a}_{-i}$,

$$\frac{\partial U^i}{\partial a_i}\left(a^*_{i,t}, a^*_{-i,t'}\right) \leqslant 0$$

and, since $U^i\left(\cdot, a^*_{-i,t'}\right)$ is strongly concave and $a^*_{i,t} < a^*_{i,t'}$, it follows that

$$\frac{\partial U^i}{\partial a_i}\left(a^*_{i,t'}, a^*_{-i,t'}\right) < 0$$

which implies, again by the Kühn-Tucker theorem, and since

$$a^*_{i,t'} \in Arg \max_{a_i \in \left[0, \overline{a}_{i,t'}\right]} U^i\left(a_i, a^*_{-i,t'}\right)$$

that $a^*_{i,t'} = 0$, and hence that $\left[0, a^*_{i,t'}\right) = \varnothing$, contradicting the assumption that

$$\begin{aligned} a^*_{i,t} &\in \left[0, \min\left\{\overline{a}_{i,t}, a^*_{i,t'}\right\}\right) \\ &\subseteq \left[0, a^*_{i,t'}\right) \end{aligned}$$

Similarly for condition (4). Suppose that for some $t, t' \in \mathcal{T}$ we have that $a^*_{-i,t} \in \left[\widetilde{a}_{-i}, \overline{\overline{a}}_{-i}\right]$ and $a^*_{-i,t'} \geqslant a^*_{-i,t}$, yet $a^*_{i,t} < \min\left\{\overline{a}_{i,t}, a^*_{i,t'}\right\}$. By Nash rationalizability,

$$a^*_{i,t} \in Arg \max_{a_i \in [0, \overline{a}_{i,t}]} U^i\left(a_i, a^*_{-i,t}\right)$$

and then, by the Kühn-Tucker theorem, since $a^*_{i,t} < \overline{a}_{i,t}$, it must be true that

$$\frac{\partial U^i}{\partial a_i}\left(a^*_{i,t}, a^*_{-i,t}\right) \leqslant 0$$

and then, since U^i exhibits strategic substitutability on $\left[\widetilde{a}_{-i}, \overline{\overline{a}}_{-i}\right]$ and $a^*_{i,t'} \geqslant a^*_{-i,t} \geqslant \widetilde{a}_{-i}$,

$$\frac{\partial U^i}{\partial a_i}\left(a^*_{i,t}, a^*_{-i,t'}\right) \leqslant 0$$

and, since $U^i\left(\cdot, a^*_{-i,t'}\right)$ is strongly concave and $a^*_{i,t} < a^*_{i,t'}$,

$$\frac{\partial U^i}{\partial a_i}\left(a^*_{i,t'}, a^*_{-i,t'}\right) < 0$$

which implies, again by the Kühn-Tucker theorem, and since

$$a_{i,t'}^{*} \in Arg \max_{a_i \in \left[0, \overline{a}_{i,t'}\right]} U^{i}\left(a_i, a_{-i,t'}^{*}\right)$$

that $a_{i,t'}^{*} = 0$, and hence that $\left[0, a_{i,t'}^{*}\right) = \varnothing$, contradicting the assumption that

$$\begin{aligned} a_{i,t}^{*} &\in \left[0, \min\left\{\overline{a}_{i,t}, a_{i,t'}^{*}\right\}\right) \\ &\subseteq \left[0, a_{i,t'}^{*}\right) \end{aligned}$$

Sufficiency: Fix $i \in \mathcal{I}$. Construct $\mathcal{T}^{i}$ as follows:

- $\tau_1^i = \{1\}$
- For $t \in \{2, ..., T\}$,

$$\tau_t^i = \begin{cases} \varnothing \ \text{ if } \ (\exists t' \in \{1, ..., t-1\}) : \begin{cases} a_{-i,t}^{*} = a_{-i,t'}^{*} \\ a_{i,t}^{*} \in [0, \overline{a}_{i,t'}] \\ a_{i,t'}^{*} \in [0, \overline{a}_{i,t}] \end{cases} \\ \{t\} \ \text{ otherwise} \end{cases}$$

- $\mathcal{T}_0^i = \bigcup_{t=1}^{T} \tau_t^i$
- $\mathcal{T}_1^i = \left\{ t \in \mathcal{T}_0^i \middle| \left(\exists t' \in \mathcal{T}_0^i \backslash \{t\}\right) : a_{-i,t'}^{*} = a_{-i,t}^{*} \right\}$
- $\mathcal{T}_2^i = \left\{ t \in \mathcal{T}_1^i \middle| \left(\forall t' \in \mathcal{T}_1^i : a_{-i,t'}^{*} = a_{-i,t}^{*}\right) : \overline{a}_{i,t'} \leqslant \overline{a}_{i,t} \right\}$
- $\mathcal{T}^i = \left(\mathcal{T}_0^i \backslash \mathcal{T}_1^i\right) \cup \mathcal{T}_2^i$

It follows by construction that if $t \in \mathcal{T}_0^i \backslash \mathcal{T}_1^i$, then

$$\left(\forall t' \in \mathcal{T}^i \backslash \{t\}\right) : a_{-i,t'}^{*} \neq a_{-i,t}^{*}$$

since $\mathcal{T}^i \subseteq \mathcal{T}_0^i$. Moreover, if $t, t' \in \mathcal{T}_2^i$, $t \neq t'$, and $a_{-i,t}^{*} = a_{-i,t'}^{*}$, then, by definition, $\overline{a}_{-i,t} = \overline{a}_{-i,t'}$, which implies that $a_{i,t}^{*} \in [0, \overline{a}_{i,t'}]$ and $a_{i,t'}^{*} \in [0, \overline{a}_{i,t}]$. But this is impossible since, assuming without loss of generality that $t < t'$, then $\tau_{t'}^i = \varnothing$ and $t' \notin \mathcal{T}_0^i$.[22] Therefore,

$$\left(\forall t, t' \in \mathcal{T}^i : t \neq t'\right) : a_{-i,t}^{*} \neq a_{-i,t'}^{*}$$

Reorder $\mathcal{T}^i$ as

$$\left\{t_1^i, t_2^i, ..., t_{T^i}^i\right\}$$

[22] This shows the meaning of the set $\mathcal{T}_0^i$: it contains the indices of the observations that are non-redundant from the point of view of agent i, since he does not care about $\overline{a}_{-i}$. $\mathcal{T}^i$ is the set that contains the indices of all the observations that are relevant for i and contain the most information, in terms of revealed preferences, because they have the largest domains.

where $T^i = \#\mathcal{T}^i$, so that

$$\left(\forall n \in \left\{1,...,T^i - 1\right\}\right) : a^*_{-i,t^i_n} < a^*_{-i,t^i_{n+1}}$$

I will consider only the case when $a^*_{-i,t^i_1} \leqslant \widetilde{a}_{-i} \leqslant a^*_{-i,t^i_{T^i}}$, since the other cases derive easily from this one. Construct the mapping

$$\vartheta^i : \left(\bigcup_{n=1}^{T^i} \left\{a^*_{-i,t^i_n}\right\}\right) \cup \{\widetilde{a}_{-i}\} \longrightarrow A^i$$

(recursively) as follows:

- $\vartheta^i\left(a^*_{-i,t^i_1}\right) = a^*_{i,t^i_1}$
- $\vartheta^i\left(a^*_{-i,t^i_{T^i}}\right) = a^*_{i,t^i_{T^i}}$
- For $n > 1$ such that $a^*_{-i,t^i_n} \in [0, \widetilde{a}_{-i})$,

$$\vartheta^i\left(a^*_{-i,t^i_n}\right) = \max\left\{\max_{n'<n}\left\{\vartheta^i\left(a^*_{-i,t^i_{n'}}\right)\right\}, a^*_{i,t^i_n}\right\}$$

- For $n < T^i$ such that $a^*_{-i,t^i_n} \in \left(\widetilde{a}_{-i}, \overline{\overline{a}}_{-i}\right]$,

$$\vartheta^i\left(a^*_{-i,t^i_n}\right) = \max\left\{\max_{n'>n}\left\{\vartheta^i\left(a^*_{-i,t^i_{n'}}\right)\right\}, a^*_{i,t^i_n}\right\}$$

- For $1 < n < T^i$ such that $a^*_{-i,t^i_n} = \widetilde{a}_{-i}$,

$$\vartheta^i\left(a^*_{-i,t^i_n}\right) = \max\left\{\max_{n'\neq n}\left\{\vartheta^i\left(a^*_{-i,t^i_{n'}}\right)\right\}, a^*_{i,t^i_n}\right\}$$

- $\vartheta^i\left(\widetilde{a}_{-i}\right) = \max_n\left\{\vartheta^i\left(a^*_{-i,t^i_n}\right)\right\}$.

This mapping is single-valued and has the property that, for each $n, n' \in \left\{1,...,T^i\right\}$,

$$\begin{array}{rcl}
\left.\begin{array}{c} a^*_{-i,t^i_n} \in [0, \widetilde{a}_{-i}] \\ a^*_{-i,t^i_{n'}} \leqslant a^*_{-i,t^i_n} \end{array}\right\} & \Longrightarrow & \vartheta^i\left(a^*_{-i,t^i_{n'}}\right) \leqslant \vartheta^i\left(a^*_{-i,t^i_n}\right) \\
\left.\begin{array}{c} a^*_{-i,t^i_n} \in \left[\widetilde{a}_{-i}, \overline{\overline{a}}_{-i}\right] \\ a^*_{-i,t^i_{n'}} \geqslant a^*_{-i,t^i_n} \end{array}\right\} & \Longrightarrow & \vartheta^i\left(a^*_{-i,t^i_{n'}}\right) \leqslant \vartheta^i\left(a^*_{-i,t^i_n}\right) \\
\vartheta^i\left(a^*_{-i,t^i_n}\right) & \leqslant & \vartheta^i\left(\widetilde{a}_{-i}\right)
\end{array}$$

Let $\mathbf{l}^i \subseteq A^{-i} \times A^i$ be the linear interpolation of the following finite sequence on $\mathbb{R}^2$,

$$\left\{\left(0, \vartheta^i\left(a^*_{-i,t^i_1}\right)\right), \left(a^*_{-i,t^i_1}, \vartheta^i\left(a^*_{-i,t^i_1}\right)\right), ..., \left(\widetilde{a}_{-i}, \vartheta^i\left(\widetilde{a}_{-i}\right)\right), \right.$$
$$\left. ..., \left(a^*_{-i,t^i_{T^i}}, \vartheta^i\left(a^*_{-i,t^i_{T^i}}\right)\right), \left(\overline{\overline{a}}_{-i}, \vartheta^i\left(a^*_{-i,t^i_{T^i}}\right)\right)\right\}$$

the first components of whose elements are ordered increasingly. Let $\Phi^i : A^{-i} \longrightarrow A^i$ be the function whose graph is $\mathbf{l}^i$. This function is well defined, since

$$0 \leqslant a^*_{-i,t^i_1} \leqslant \widetilde{a}_{-i} \leqslant a^*_{-i,t^i_{T^i}} \leqslant \overline{\overline{a}}_{-i}$$

and ϑ^i is single-valued. Moreover, Φ^i has the following properties: it is continuous, nondecreasing on $[0, \widetilde{a}_{-i}]$ and nonincreasing on $\left[\widetilde{a}_{-i}, \overline{\overline{a}}_{-i}\right]$, and for each $t \in \mathcal{T}^i$, $\Phi^i\left(a^*_{-i,t}\right) \geqslant a^*_{i,t}$.

Define the function $U^i : A^i \times A^{-i} \longrightarrow \mathbb{R}$ by

$$U^i\left(a_i, a_{-i}\right) = -\left(a_i - \Phi^i\left(a_{-i}\right)\right)^2$$

I now show that $\left(U^i\right)_{i \in \mathcal{I}}$ Nash-rationalizes the data set. It is immediate that for each $i \in \mathcal{I}$, U^i is continuous and $\forall a_{-i} \in A^{-i}$, $U^i\left(\cdot, a_{-i}\right)$ is differentiable and strongly concave. Then, I only need to show that

$$\left(\forall t \in \mathcal{T}\right)\left(\forall i \in \mathcal{I}\right) : a^*_{i,t} \in Arg \max_{a_i \in [0, \overline{a}_{i,t}]} U^i\left(a_i, a^*_{-i,t}\right)$$

Fix $t \in \mathcal{T}$ and $i \in \mathcal{I}$.

Suppose first that $t \in \mathcal{T}^i$. If $\Phi^i\left(a^*_{-i,t}\right) = a^*_{i,t}$, the result is obvious. Else, if $a^*_{-i,t} \in [0, \widetilde{a}_{-i})$, $\Phi^i\left(a^*_{-i,t}\right) > a^*_{i,t}$ implies that for some $t' \in \mathcal{T}^i$, $a^*_{-i,t'} < a^*_{-i,t}$ and $a^*_{i,t'} > a^*_{i,t}$. Then, by condition (3), $a^*_{i,t} = \overline{a}_{i,t} < \Phi^i\left(a^*_{-i,t}\right)$, from where

$$\left\{a^*_{i,t}\right\} = Arg \max_{a_i \in [0, \overline{a}_{i,t}]} U^i\left(a_i, a^*_{-i,t}\right)$$

If, on the other hand, $a^*_{-i,t} \in \left(\widetilde{a}_{-i}, \overline{\overline{a}}_{-i}\right]$, $\Phi^i\left(a^*_{-i,t}\right) > a^*_{i,t}$ implies that for some $t' \in \mathcal{T}^i$, $a^*_{-i,t'} > a^*_{-i,t}$ and $a^*_{i,t'} > a^*_{i,t}$. Then, by condition (4), $a^*_{i,t} = \overline{a}_{i,t} < \Phi^i\left(a^*_{-i,t}\right)$, from where, as before,

$$\left\{a^*_{i,t}\right\} = Arg \max_{a_i \in [0, \overline{a}_{i,t}]} U^i\left(a_i, a^*_{-i,t}\right)$$

Finally, if $a^*_{-i,t} = \widetilde{a}_{-i}$, $\Phi^i\left(a^*_{-i,t}\right) > a^*_{i,t}$ implies that either for some $t' \in \mathcal{T}^i$, $a^*_{-i,t'} < a^*_{-i,t}$ and $a^*_{i,t'} > a^*_{i,t}$ or for some $t' \in \mathcal{T}^i$, $a^*_{-i,t'} > a^*_{-i,t}$ and $a^*_{i,t'} > a^*_{i,t}$, and the results follows according to the corresponding previous case.

Suppose now that $t \in \mathcal{T}_0^i \backslash \mathcal{T}^i = \mathcal{T}_1^i \backslash \mathcal{T}_2^i$.[23] By construction, $\exists t' \in \mathcal{T}_1^i$ such that $a^*_{-i,t'} = a^*_{-i,t}$ and $\overline{a}_{i,t'} > \overline{a}_{i,t}$. Let

$$t'_t \in Arg \max_{t' \in \mathcal{T}_i^1} \left\{ \overline{a}_{i,t'} \,|\, a^*_{-i,t'} = a^*_{-i,t} \right\}$$

By construction, $t'_t \in \mathcal{T}_2^i \subseteq \mathcal{T}^i$, $\overline{a}_{i,t'_t} > \overline{a}_{i,t}$ and $a^*_{-i,t'_t} = a^*_{-i,t}$, so that $\forall a_i \in A^i$:

$$\begin{aligned} U^i\left(a_i, a^*_{-i,t}\right) &= U^i\left(a_i, a^*_{-i,t'_t}\right) \\ &= -\left(a_i - \Phi^i\left(a^*_{-i,t'_t}\right)\right)^2 \end{aligned}$$

whereas $a^*_{i,t} \in [0, \overline{a}_{i,t}] \subset \left[0, \overline{a}_{i,t'_t}\right]$, so that if $a^*_{-i,t'_t} \in [0, \overline{a}_{i,t}]$, then, by condition (1), $a^*_{i,t} = a^*_{i,t'_t}$ and, therefore,

$$\left\{a^*_{i,t}\right\} = Arg \max_{a_i \in [0, \overline{a}_{i,t}]} U^i\left(a_i, a^*_{-i,t}\right)$$

follows from the previous case. If, alternatively, $a^*_{-i,t'_t} \notin [0, \overline{a}_{i,t}]$, then, by condition (2), $a^*_{i,t} = \overline{a}_{i,t}$ and, since $\overline{a}_{i,t} < a^*_{i,t'_t} \leqslant \Phi^i\left(a^*_{-i,t'_t}\right)$, again

$$\left\{a^*_{i,t}\right\} = Arg \max_{a_i \in [0, \overline{a}_{i,t}]} U^i\left(a_i, a^*_{-i,t}\right)$$

Finally, consider $t \in \mathcal{T} \backslash \mathcal{T}_0^i$. Again, $\exists t' \in \{1, ..., t-1\}$ such that $a^*_{-i,t} = a^*_{-i,t'}$, $a^*_{i,t} \in [0, \overline{a}_{i,t'}]$ and $a^*_{i,t'} \in [0, \overline{a}_{i,t}]$. Let

$$t'_t = \min\left\{ t' \in \{1, ..., t-1\} \,|\, a^*_{-i,t} = a^*_{-i,t'}, a^*_{i,t} \in [0, \overline{a}_{i,t'}], a^*_{i,t'} \in [0, \overline{a}_{i,t}] \right\}$$

By construction, $t'_t \in \mathcal{T}_0^i$ and then, by the previous two cases, and condition (1),

$$\begin{aligned} \left\{a^*_{i,t}\right\} &= \left\{a^*_{i,t'_t}\right\} \\ &= Arg \max_{a_i \in \left[0, \overline{a}_{i,t'_t}\right]} U^i\left(a_i, a^*_{-i,t'_t}\right) \\ &= Arg \max_{a_i \in [0, \overline{a}_{i,t}]} U^i\left(a_i, a^*_{-i,t}\right) \end{aligned}$$

[23] This, because

$$\begin{aligned} \mathcal{T}_0^i \backslash \mathcal{T}^i &= \mathcal{T}_0^i \backslash \left(\left(\mathcal{T}_0^i \backslash \mathcal{T}_1^i \right) \cup \mathcal{T}_2^i \right) \\ &= \mathcal{T}_0^i \cap \left(\left(\mathcal{T}_0^i \backslash \mathcal{T}_1^i \right) \cup \mathcal{T}_2^i \right)^c \\ &= \mathcal{T}_0^i \cap \left(\left(\mathcal{T}_0^i \backslash \mathcal{T}_1^i \right)^c \cap \left(\mathcal{T}_2^i \right)^c \right) \\ &= \mathcal{T}_0^i \cap \left(\left(\mathcal{T}_0^i \cap \left(\mathcal{T}_1^i \right)^c \right)^c \cap \left(\mathcal{T}_2^i \right)^c \right) \\ &= \mathcal{T}_0^i \cap \left(\left(\mathcal{T}_0^i \right)^c \cup \mathcal{T}_1^i \right) \cap \left(\mathcal{T}_2^i \right)^c \\ &= \mathcal{T}_1^i \cap \left(\mathcal{T}_2^i \right)^c \\ &= \mathcal{T}_1^i \backslash \mathcal{T}_2^i \end{aligned}$$

where all complements are taken relatively to $\mathcal{T}_0^i$.

All that remains to show is that each U^i exhibits strategic complementarity on $[0, \widetilde{a}_{-i}]$ and strategic substitutability on $\left[\widetilde{a}_{-i}, \overline{\overline{a}}_{-i}\right]$. Fix $i \in \mathcal{I}$ and $a_i' \in A^i$. By construction,

$$\frac{\partial U^i}{\partial a_i}\left(a_i', \cdot\right) = -2\left(a_i' - \Phi^i\left(\cdot\right)\right)$$

which is nondecreasing on $[0, \widetilde{a}_{-i}]$ and nonincreasing on $\left[\widetilde{a}_{-i}, \overline{\overline{a}}_{-i}\right]$. ■

Hence, when the hypotheses of strategic complementarity and substitutability can be plausibly assumed, they strengthen the test of corollary 3, as they can themselves be tested. It is straightforward to see that the restrictions derived from these extra assumptions are not "zero measure" and hence the conclusion of subsection 3.3 does not follow here. It is also simple to see that one can impose other structures of complementarity and substitutability, in which case suitably modified restrictions will arise.

5.2 Stronger testable restrictions:

In the results obtained in section 3, players make very fine distinctions regarding the actions of their opponents. They do care, for example, for the exact value of the action of each opponent and treat their opponents distinctly, with careful consideration for who they are. Although this may be acceptable in many situations (for example if the actions of others are not physically comparable, or if one opponent is a good friend, or a partner in a joint venture of one firm, and the other opponent is an enemy, or the competitor of the firm) it may also be too extreme in other cases: individual actions may be physically comparable, and players may care about the values of the actions of the opponents, regardless of the names that those actions have attached (effectively treating their opponents as mutual substitutes,) or they may care about the actions of only some, or none, of their opponents, or about just an aggregate of their actions. I now show that these types of assumptions may yield stronger versions of the conditions of corollary 3.

5.2.1 Aggregation:

I first consider the case in which each individual only cares about his own action and an aggregate of his opponents choices. I assume for the remainder of this subsection that $\forall i \in \mathcal{I}$, $\overline{\overline{a}}_i = \overline{\overline{a}}$. Denote $A = \left[0, \overline{\overline{a}}\right]$ and $A^- = \left[0, \overline{\overline{a}}\right]^{I-1}$. I consider general aggregators, which are defined as follows:

Definition 7 *A function $\sigma : A^- \longrightarrow S$, where $S \subseteq R^n$, $n \in \mathbb{N}$, is an aggregator for $U : A \times A^- \longrightarrow \mathbb{R}$ if for each $a_i, a_i' \in A^-$*

$$\sigma\left(a_i\right) = \sigma\left(a_i'\right) \Longrightarrow \left(\forall a \in A\right) : U\left(a, a_i\right) = U\left(a, a_i'\right)$$

Now, suppose that one has theoretical reasons to assume that each individual $i \in \mathcal{I}$ only cares about his own choice, and an aggregate σ^i of the choices of his opponents, where $\sigma^i : A^- \longrightarrow S^i$, for $S^i \subseteq R^{n_i}$, $n_i \in \mathbb{N}$. Then, the definition of Nash-rationalizability must be modified accordingly.

Definition 8 *Let* $\boldsymbol{\sigma}=\left(\sigma^i : \mathbb{R}_+^{I-1} \longrightarrow S^i\right)_{i\in\mathcal{I}}$ *be a profile of continuous aggregators. A data set*

$$\left(\left(a_{i,t}^*, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)_{t=1}^T$$

is Nash-rationalizable with $\boldsymbol{\sigma}$*-aggregation if there exists* $\left(U^i\right)_{i\in\mathcal{I}}$ *that Nash-rationalizes it, such that for each* $i\in\mathcal{I}$, σ^i *is an aggregator for* U^i*.*

Intuition suggests that all the restrictions of Nash behavior in this context should be modified versions of the restrictions of corollary 3, strengthened so that, for each $i\in\mathcal{I}$, it is not the profile of actions of i's opponents, but its aggregate according to σ^i that conditions them. The following theorem confirms this intuition.

Theorem 5 *Let* $\boldsymbol{\sigma}=\left(\sigma^i : \mathbb{R}_+^{I-1} \longrightarrow S^i\right)_{i\in\mathcal{I}}$ *be a profile of continuous aggregators. A data set*

$$\left(\left(a_{i,t}^*, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)_{t=1}^T$$

is Nash-rationalizable with $\boldsymbol{\sigma}$*-aggregation if, and only if, for each* $t,t'\in\mathcal{T}$ *and each* $i\in\mathcal{I}$ *:*

1.
$$\left.\begin{array}{c}\sigma^i\left(a_{-i,t}^*\right)=\sigma^i\left(a_{-i,t'}^*\right)\\ a_{i,t}^*\in[0,\overline{a}_{i,t'}]\\ a_{i,t'}^*\in[0,\overline{a}_{i,t}]\end{array}\right\}\Longrightarrow a_{i,t}^*=a_{i,t'}^*$$

2.
$$\left.\begin{array}{c}\sigma^i\left(a_{-i,t}^*\right)=\sigma^i\left(a_{-i,t'}^*\right)\\ a_{i,t}^*\notin[0,\overline{a}_{i,t'}]\end{array}\right\}\Longrightarrow a_{i,t'}^*=\overline{a}_{i,t'}$$

Proof. Necessity: Suppose not. Let $\left(U^i\right)_{i\in\mathcal{I}}$ Nash-rationalize the data, assuming that $\forall i\in\mathcal{I}, U^i$ satisfies that

$$\left(\forall a_i, a_i'\in A^- : \sigma^i\left(a_i\right)=\sigma^i\left(a_i'\right)\right)\left(\forall a\in A\right) : U^i\left(a,a_i\right)=U^i\left(a,a_i'\right)$$

Suppose that $\exists i\in\mathcal{I}$ and $\exists t,t'\in\mathcal{T}$ such that $\sigma^i\left(a_{-i,t}^*\right)=\sigma^i\left(a_{-i,t'}^*\right)$, $a_{i,t}^*\in[0,\overline{a}_{i,t'}]$, $a_{i,t'}^*\in[0,\overline{a}_{i,t}]$ and $a_{i,t}^*\neq a_{i,t'}^*$. Without loss of generality, assume that $U^i\left(a_{i,t}^*, a_{-i,t}^*\right)\leqslant U^i\left(a_{i,t'}^*, a_{-i,t'}^*\right)$. Since $\sigma^i\left(a_{-i,t}^*\right)=\sigma^i\left(a_{-i,t'}^*\right)$, the latter implies that $U^i\left(a_{i,t}^*, a_{-i,t}^*\right)\leqslant U^i\left(a_{i,t'}^*, a_{-i,t}^*\right)$. Let $a_i=\frac{1}{2}\left(a_{i,t}^*+a_{i,t'}^*\right)$. Clearly, $a_i\in[0,\overline{a}_{i,t}]$ whereas by strong concavity $U^i\left(a_i, a_{-i,t}^*\right)>U^i\left(a_{i,t}^*, a_{-i,t}^*\right)$, contradicting the fact that $\left(U^i\right)_{i\in\mathcal{I}}$ Nash-rationalizes the data set. This proves condition (1).

Now, for condition (2), suppose that $\exists i\in\mathcal{I}$ and $\exists t,t'\in\mathcal{T}$ such that $\sigma^i\left(a_{-i,t}^*\right)=\sigma^i\left(a_{-i,t'}^*\right)$, $a_{i,t}^*\notin[0,\overline{a}_{i,t'}]$ and $a_{i,t'}^*<\overline{a}_{i,t'}$. Then, it follows that $\overline{a}_{i,t'}<\overline{a}_{i,t}$. Also, since $a_{i,t'}^*\in[0,\overline{a}_{i,t'}]\subset[0,\overline{a}_{i,t}]$, it follows that if $\left(U^i\right)_{i\in\mathcal{I}}$ Nash-rationalizes the data set, then $U^i\left(a_{i,t}^*, a_{-i,t}^*\right)\geqslant U^i\left(a_{i,t'}^*, a_{-i,t}^*\right)$. Moreover, since

$\sigma^i\left(a^*_{-i,t}\right)=\sigma^i\left(a^*_{-i,t'}\right)$, the latter implies that $U^i\left(a^*_{i,t},a^*_{-i,t'}\right)\geqslant U^i\left(a^*_{i,t'},a^*_{-i,t'}\right)$ and, then, by strong concavity,

$$(\forall\lambda\in(0,1)):U^i\left(\lambda a^*_{i,t}+(1-\lambda)\,a^*_{i,t'},a^*_{-i,t'}\right)>U^i\left(a^*_{i,t'},a^*_{-i,t'}\right)$$

whereas for λ close enough to 0, $\lambda a^*_{i,t}+(1-\lambda)\,a^*_{i,t'}\in[0,\overline{a}_{i,t'}]$, contradicting the fact that $\left(U^i\right)_{i\in\mathcal{I}}$ Nash-rationalizes the data.

Sufficiency: Fix $i\in\mathcal{I}$. Define the set $\mathcal{T}^i$ as follows:

- $\tau^i_1=\{1\}$
- For $t\in\{2,...,T\}$,

$$\tau^i_t=\begin{cases}\varnothing\ \text{ if }\ (\exists t'\in\{1,...,t-1\}):\begin{cases}\sigma^i\left(a^*_{-i,t}\right)=\sigma^i\left(a^*_{-i,t'}\right)\\ a^*_{i,t}\in[0,\overline{a}_{i,t'}]\\ a^*_{i,t'}\in[0,\overline{a}_{i,t}]\end{cases}\\ \{t\}\quad\text{otherwise}\end{cases}$$

- $\mathcal{T}^i_0=\bigcup\limits_{t=1}^{T}\tau^i_t$
- $\mathcal{T}^i_1=\left\{t\in\mathcal{T}^i_0\middle|\left(\exists t'\in\mathcal{T}^i_0\backslash\{t\}\right):\sigma^i\left(a^*_{-i,t'}\right)=\sigma^i\left(a^*_{-i,t}\right)\right\}$
- $\mathcal{T}^i_2=\left\{t\in\mathcal{T}^i_1\middle|\left(\forall t'\in\mathcal{T}^i_1:\sigma^i\left(a^*_{-i,t'}\right)=\sigma^i\left(a^*_{-i,t}\right)\right):\overline{a}_{i,t'}\leqslant\overline{a}_{i,t}\right\}$
- $\mathcal{T}^i=\left(\mathcal{T}^i_0\backslash\mathcal{T}^i_1\right)\cup\mathcal{T}^i_2$

It follows by construction that if $t\in\mathcal{T}^i_0\backslash\mathcal{T}^i_1$, then

$$\left(\forall t'\in\mathcal{T}^i\backslash\{t\}\right):\sigma^i\left(a^*_{-i,t'}\right)\neq\sigma^i\left(a^*_{-i,t}\right)$$

since $\mathcal{T}^i\subseteq\mathcal{T}^i_0$. Moreover, if $t,t'\in\mathcal{T}^i_2$, $t\neq t'$, are such that $\sigma^i\left(a^*_{-i,t}\right)=\sigma^i\left(a^*_{-i,t'}\right)$, then, by construction, $\overline{a}_{-i,t}=\overline{a}_{-i,t'}$, and, therefore $a^*_{i,t}\in[0,\overline{a}_{i,t'}]$ and $a^*_{i,t'}\in[0,\overline{a}_{i,t}]$. But this is impossible since, assuming without loss of generality that $t<t'$, then $\tau^i_{t'}=\varnothing$ and $t'\notin\mathcal{T}^i_0$. Then, we conclude that

$$\left(\forall t,t'\in\mathcal{T}^i:t\neq t'\right):\sigma^i\left(a^*_{-i,t}\right)\neq\sigma^i\left(a^*_{-i,t'}\right)$$

Define $C^i=\left\{\sigma^i\left(a^*_{-i,t}\right)\right\}_{t\in\mathcal{T}^i}\subseteq\sigma^i\left[A^-\right]$. By the previous result, the function $\phi^i:C^i\longrightarrow A$, defined by $\theta^i\left(\sigma^i\left(a^*_{-i,t}\right)\right)=a^*_{i,t}$, for each $t\in\mathcal{T}^i$, is well defined.[24]

[24]By using this function, I am giving up the possibility of ensuring a local uniqueness result analogous to theorem 2. If such a result is wanted, one can reason as follows: since $\#\mathcal{T}^i\leqslant T<\infty$, we have that

$$\overline{\varepsilon}_i:=\min_{t,t'\in\mathcal{T}^i:t\neq t'}\left(\left\|\sigma^i\left(a^*_{-i,t}\right)-\sigma^i\left(a^*_{-i,t'}\right)\right\|\right)>0$$

Fix $\varepsilon_i\in(0,\overline{\varepsilon}_i)$. Since σ^i is continuous and $\#\mathcal{T}^i<\infty$, $\exists\delta_i\in\mathbb{R}_{++}$ such that

$$\left(\forall t\in T^i\right)(\forall a_{-i}\in A_-):\left\|a_{-i}-a^*_{-i,t}\right\|\leqslant\delta_i\Longrightarrow\left\|\sigma^i\left(a_{-i}\right)-\sigma^i\left(a^*_{-i,t}\right)\right\|<\varepsilon_i$$

Since C^i is closed, and ϕ^i is (trivially) continuous and bounded, it follows from Tietze's extension theorem that there exists a continuous extension of ϕ^i to the whole of $\sigma^i[A^-]$. Let $\Phi^i : \sigma^i[A^-] \longrightarrow A$ be one such extension.

Define now $U^i : A \times A^- \longrightarrow \mathbb{R}$ by

$$U^i(a_i, a_{-i}) = -\left(a_i - \Phi^i\left(\sigma^i(a_{-i})\right)\right)^2$$

Clearly, U^i is continuous and satisfies that

$$\left(\forall a_i, a_i' \in A^- : \sigma^i(a_i) = \sigma^i(a_i')\right)(\forall a \in A) : U^i(a, a_i) = U^i(a, a_i')$$

I now only have to show that, so defined, $\left(U^i\right)_{i\in\mathcal{I}}$ Nash-rationalizes the data set. It is immediate that $\forall a_{-i} \in A^-$, the function $U^i(\cdot, a_{-i})$ is differentiable and strongly concave. In order to show that

$$(\forall t \in \mathcal{T}) : \left(a^*_{i,t}\right)_{i\in\mathcal{I}} \in N\left(\left([0, \overline{a}_{i,t}], U^i_{\overline{a}_t}\right)_{i\in\mathcal{I}}\right)$$

I have to show that

$$(\forall t \in \mathcal{T})(\forall i \in \mathcal{I}) : a^*_{i,t} \in Arg \max_{a_i \in [0, \overline{a}_{i,t}]} U^i\left(a_i, a^*_{-i,t}\right)$$

Hence, fix $t \in \mathcal{T}$ and $i \in \mathcal{I}$.

Define the set

$$C^i := \bigcup_{t\in\mathcal{T}^i} \overline{B_{\frac{\delta_i}{2}}\left(a^*_{-i,t}\right)} \cap A_-$$

and the function $\phi^i : C^i \longrightarrow A_i$ by

$$\phi^i(a_{-i}) := \min\left\{Arg \max_{a_i \in \left\{a^*_{i,t}\right\}_{t\in\mathcal{T}^i}} \left(\min_{t'\in\mathcal{T}^i}\left(\left(\left\|\sigma^i(a_{-i}) - \sigma^i\left(a^*_{-i,t'}\right)\right\| - \frac{\varepsilon_i}{2}\right)\left(a_i - a^*_{i,t'}\right)^2\right)\right)\right\}$$

One first shows that

$$(\forall t \in \mathcal{T}^i)\left(\forall a_{-i} \in \overline{B_{\frac{\delta_i}{2}}\left(a^*_{-i,t}\right)} \cap A_-\right) : \phi^i(a_{-i}) = a^*_{i,t}$$

To see this, let $t \in \mathcal{T}^i$ and $a_{-i} \in \overline{B_{\frac{\delta_i}{2}}\left(a^*_{-i,t}\right)} \cap A_-$. Consider, for each $a_i \in \left\{a^*_{i,t''}\right\}_{t''\in\mathcal{T}^i}$, the problem

$$\min_{t'\in\mathcal{T}^i}\left(\left(\left\|\sigma^i(a_{-i}) - \sigma^i\left(a^*_{-i,t'}\right)\right\| - \frac{\varepsilon_i}{2}\right)\left(a_i - a^*_{i,t'}\right)^2\right)$$

By construction, definition of δ_i and ε_i and triangle inequality

$$(\forall t' \in \mathcal{T}^i \setminus \{t\}) : \left\|\sigma^i(a_{-i}) - \sigma^i\left(a^*_{-i,t'}\right)\right\| > \frac{\varepsilon_i}{2}$$

whereas

$$\left\|\sigma^i(a_{-i}) - \sigma^i\left(a^*_{-i,t}\right)\right\| \leqslant \frac{\varepsilon_i}{2}$$

which suffices to imply that

$$\min_{t'\in\mathcal{T}^i}\left(\left(\left\|a_{-i} - a^*_{-i,t'}\right\| - \frac{\varepsilon_i}{2}\right)\left(a_i - a^*_{i,t'}\right)^2\right) = \left(\left\|a_{-i} - a^*_{-i,t}\right\| - \frac{\varepsilon_i}{2}\right)\left(a_i - a^*_{i,t}\right)^2$$

and therefore that $\phi^i(a_{-i}) = a^*_{i,t}$. From here on, the proof carries on.

Suppose first that $t \in \mathcal{T}^i$. By construction,

$$\begin{array}{rcl} U^i\left(a_i, a^*_{-i,t}\right) & = & -\left(a_i - \Phi^i\left(\sigma^i\left(a^*_{-i,t}\right)\right)\right)^2 \\ & = & -\left(a_i - \phi^i\left(\sigma^i\left(a^*_{-i,t}\right)\right)\right)^2 \\ & = & -\left(a_i - a^*_{i,t}\right)^2 \end{array}$$

from where the result is obvious.

Secondly, consider $t \in \mathcal{T}^i_0 \backslash \mathcal{T}^i = \mathcal{T}^i_1 \backslash \mathcal{T}^i_2$.[25] Then, by construction, $\exists t' \in \mathcal{T}^i_1$ such that $\sigma^i\left(a^*_{-i,t'}\right) = \sigma^i\left(a^*_{-i,t}\right)$ and $\overline{a}_{i,t'} > \overline{a}_{i,t}$. Let

$$t'_t \in Arg \max_{t' \in \mathcal{T}^1_i} \left\{ \overline{a}_{i,t'} \middle| \, \sigma^i\left(a^*_{-i,t'}\right) = \sigma^i\left(a^*_{-i,t}\right) \right\}$$

By construction, $t'_t \in \mathcal{T}^i_2 \subseteq \mathcal{T}^i$, $\overline{a}_{i,t'_t} > \overline{a}_{i,t}$ and $\sigma^i\left(a^*_{-i,t'_t}\right) = \sigma^i\left(a^*_{-i,t}\right)$, so that $\forall a_i \in A:$

$$\begin{array}{rcl} U^i\left(a_i, a^*_{-i,t}\right) & = & -\left(a_i - \Phi^i\left(\sigma^i\left(a^*_{-i,t}\right)\right)\right)^2 \\ & = & -\left(a_i - \Phi^i\left(\sigma^i\left(a^*_{-i,t'_t}\right)\right)\right)^2 \\ & = & -\left(a_i - a^*_{i,t'_t}\right)^2 \end{array}$$

whereas $a^*_{i,t} \in [0, \overline{a}_{i,t}] \subset \left[0, \overline{a}_{i,t'_t}\right]$, so that if $a^*_{i,t'_t} \in [0, \overline{a}_{i,t}]$, then by condition (1) of the theorem $a^*_{i,t} = a^*_{i,t'_t}$ and, therefore

$$\left\{a^*_{i,t}\right\} = Arg \max_{a_i \in [0, \overline{a}_{i,t}]} U^i\left(a_i, a^*_{-i,t}\right)$$

If, alternatively, $a^*_{i,t'_t} \notin [0, \overline{a}_{i,t}]$, then, by condition (2) of the theorem, $a^*_{i,t} = \overline{a}_{i,t}$ and, since $\overline{a}_{i,t} < a^*_{i,t'_t}$, again

$$\left\{a^*_{i,t}\right\} = Arg \max_{a_i \in [0, \overline{a}_{i,t}]} \left(a_i, a^*_{-i,t}\right)$$

Finally, suppose that $t \in \mathcal{T} \backslash \mathcal{T}^i_0$. Then, by construction, $\exists t' \in \{1, ..., t-1\}$ such that $\sigma^i\left(a^*_{-i,t'}\right) = \sigma^i\left(a^*_{-i,t}\right)$, $a^*_{i,t} \in [0, \overline{a}_{i,t'}]$ and $a^*_{i,t'} \in [0, \overline{a}_{i,t}]$. Let

$$t'_t = \min\left\{ t' \in \{1, ..., t-1\} \middle| \, \sigma^i\left(a^*_{-i,t}\right) = \sigma^i\left(a^*_{-i,t'}\right), a^*_{i,t} \in [0, \overline{a}_{i,t'}], a^*_{i,t'} \in [0, \overline{a}_{i,t}] \right\}$$

By construction, $t'_t \in \mathcal{T}^i_0$ and then, by the previous two cases and condition (1),

$$\begin{array}{rcl} \left\{a^*_{i,t}\right\} & = & \left\{a^*_{i,t'_t}\right\} \\ & = & Arg \max_{a_i \in \left[0, \overline{a}_{i,t'_t}\right]} U^i\left(a_i, a^*_{-i,t'_t}\right) \\ & = & Arg \max_{a_i \in [0, \overline{a}_{i,t}]} U^i\left(a_i, a^*_{-i,t}\right) \end{array}$$

[25] See note 23.

■

The importance of this result is that its testable restrictions need not have "zero measure." For this, it suffices that for some player $i \in \mathcal{I}$ there exist a subset of A^{-}, with positive Lebesgue measure, where the image of aggregator σ^i is constant. In such a case, the experiment of generating individual choices randomly, using uniform distributions, no longer generates rationalizable data sets with probability one. This probability now decreases as the measure of those level sets of the aggregators increases. The harshness of the test then depends on the Lebesgue measure of the level sets of the aggregators.

5.2.2 Example: Anonymity

A simple example of aggregation is the following. Consider a game in which actions taken by all players are physically comparable, and each player $i \in \mathcal{I}$ treats the other players anonymously in the sense that if $a_{-i}, a'_{-i} \in A^{-}$ are such that the only difference between the two of them is the order of their components, then for each $a \in A^i$, player i is indifferent between (a, a_{-i}) and $\left(a, a'_{-i}\right)$.

Define the set $\mathbb{O} = \left\{ b \in \mathbb{R}_{+}^{I-1} \middle| \left(\forall l \in \{2, ..., I-1\}\right) : b_{l-1} \leqslant b_l \right\}$, in which all the vectors are ordered ascendingly. Let the function $o : \mathbb{R}_{+}^{I-1} \longrightarrow \mathbb{O}$ reorder the elements of vectors in $\mathbb{R}_{+}^{I-1}$. That is,

$$a \longmapsto o(a) = \left(o_1(a), o_2(a), ..., o_{I-1}(a)\right)$$

such that

$$\left(\forall a \in \mathbb{R}_{+}^{I-1}\right)\left(\forall l \in \{2, ..., I-1\}\right) : o_{l-1}(a) \leqslant o_l(a)$$

and

$$\begin{aligned}
\left(\forall a \in \mathbb{R}_{+}^{I-1}\right)\left(\forall l \in \{1, ..., I-1\}\right)\left(\exists l' \in \{1, ..., I-1\}\right) &\quad : \quad o_l(a) = a_{l'} \\
\left(\forall a \in \mathbb{R}_{+}^{I-1}\right)\left(\forall l \in \{1, ..., I-1\}\right)\left(\exists l' \in \{1, ..., I-1\}\right) &\quad : \quad a_l = o_{l'}(a)
\end{aligned}$$

The idea of anonymity is that what each player cares about is the components of the vectors of choices of his opponents, regardless of their order. Then, define:

Definition 9 *Two vectors $a_{-i}, a'_{-i} \in A^{-}$ are anonymously equivalent if*

$$o(a_{-i}) = o\left(a'_{-i}\right)$$

Consequently, under the Nash hypothesis, individuals behave according to anonymity if their preferences satisfy the following condition.

Definition 10 *A function $U : A \times A^{-} \longrightarrow \mathbb{R}$ satisfies anonymity if o is an aggregator for U.*

Again, one must modify the definition of Nash-rationalizability so as to require that preferences satisfy anonymity. The natural definition is:

Definition 11 *A data set*

$$\left(\left(a_{i,t}^{*},\overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)_{t=1}^{T}$$

is Nash-rationalizable with anonymity (NRWA) if there exists $\left(U^{i}\right)_{i\in\mathcal{I}}$ *that Nash-rationalizes it such that* $\forall i\in\mathcal{I},U^{i}$ *satisfies anonymity.*

Given that the function o is continuous,[26] the following result follows straightforwardly from theorem 5.

Corollary 4 *A data set*

$$\left(\left(a_{i,t}^{*},\overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)_{t=1}^{T}$$

is NRWA if, and only if, $\forall t,t'\in\mathcal{T}$ *and* $\forall i\in\mathcal{I}$ *:*

1. $$\left.\begin{array}{c} o\left(a_{-i,t}^{*}\right)=o\left(a_{-i,t'}^{*}\right)\\ a_{i,t}^{*}\in\left[0,\overline{a}_{i,t'}\right]\\ a_{i,t'}^{*}\in\left[0,\overline{a}_{i,t}\right]\end{array}\right\}\Longrightarrow a_{i,t}^{*}=a_{i,t'}^{*}$$

2. $$\left.\begin{array}{c} o\left(a_{-i,t}^{*}\right)=o\left(a_{-i,t'}^{*}\right)\\ a_{i,t}^{*}\notin\left[0,\overline{a}_{i,t'}\right]\end{array}\right\}\Longrightarrow a_{i,t'}^{*}=\overline{a}_{i,t'}$$

Proof. Let $\boldsymbol{\sigma}=(o:A^{-}\longrightarrow\mathbb{O})_{i\in\mathcal{I}}$. This is a profile of continuous aggregators, and the result then follows from theorem 5 ■

It must be noticed, however, that anonymity does not increase the power of the test of corollary 3, since all level sets for the anonymous aggregator, o, are finite and, hence, have zero Lebesgue measure.

5.3 Example: Cournot competition.

As a simple example of a problem in which issues of aggregation and strategic substitutability are present, consider the following (slightly nonstandard) oligopoly problem. Consider an industry composed by a set $\mathcal{I}$ of firms. Each firm has a structural maximal production capacity $\overline{\overline{q}}$. At time t, firm i decides how much to produce $(q_{i,t}^{*})$, subject to a short-term capacity constraint $q_{i}\in\left[0,\overline{q}_{i,t}\right]\subseteq\left[0,\overline{\overline{q}}\right]$, taking as given the aggregate production of the rest of the industry $(\sum_{j\in\mathcal{I}\setminus\{i\}}q_{j,t}^{*})$, so as to solve the problem

$$\max_{q_{i}\in\left[0,\overline{q}_{i,t}\right]}\pi^{i}\left(q_{i},q_{-i,t}^{*}\right)$$

[26] Actually, uniformly continuous: since $\forall a,a'\in\mathbb{R}_{+}^{I-1}$, $\sum_{i=1}^{I-1}o_{i}\left(a\right)o_{i}\left(a'\right)\geqslant\sum_{i=1}^{I-1}a_{i}a_{i}'$, it is easy to show that $\forall a,a'\in\mathbb{R}_{+}^{I-1}$, $\|o\left(a\right)-o\left(a'\right)\|\leqslant\|a-a'\|$. Then, by letting $\delta=\varepsilon$ it follows that

$$\left(\forall\varepsilon\in\mathbb{R}_{++}\right)\left(\exists\delta\in\mathbb{R}_{++}\right)\left(\forall a\in\mathbb{R}_{+}^{I-1}\right)\left(\forall a'\in B_{\delta}\left(a\right)\cap\mathbb{R}_{+}^{I-1}\right):\left\|o\left(a\right)-o\left(a'\right)\right\|<\varepsilon$$

where $\pi^i : A \times A^- \longrightarrow \mathbb{R}$ is a profit function that satisfies the following properties:[27]

1. It is continuous,
2. $\forall q_{-i} \in A^-$, the function $\pi_i(\cdot, q_{-i}) : A \longrightarrow \mathbb{R}$ is differentiable and strongly concave,
3. $\left(\forall q_{-i}, q'_{-i} \in A^-\right)$:

$$\sum_{j\in\mathcal{I}\backslash\{i\}} q_j = \sum_{j\in\mathcal{I}\backslash\{i\}} q'_j \Longrightarrow (\forall q' \in A) : \pi^i(q, q_{-i}) = \pi^i\left(q, q'_{-i}\right)$$

4. $\left(\forall q_{-i}, q'_{-i} \in A^-\right)$:

$$\sum_{j\in\mathcal{I}\backslash\{i\}} q_j \leqslant \sum_{j\in\mathcal{I}\backslash\{i\}} q'_j \Longrightarrow (\forall q' \in A) : \frac{\partial \pi^i}{\partial q_i}(q, q_{-i}) \geqslant \frac{\partial \pi^{i\,i}}{\partial q_i}\left(q, q'_{-i}\right)$$

A data set is a finite time series of observed productions and short term production capacities, $\left(\left(q^*_{i,t}, \overline{q}_{i,t}\right)_{i\in\mathcal{I}}\right)_{t=1}^{T}$. It follows from combining theorems 5 and 4 that there exists a profile of profit functions $\left(\pi^i\right)_{i\in\mathcal{I}}$ such that $\forall i \in \mathcal{I}$, π^i satisfies the above conditions and $\forall t \in \mathcal{T}$, $(q_{i,t})_{i\in\mathcal{I}}$ is a Nash-Cournot equilibrium of the market given capacity constraints $\left(\overline{q}_{i,t}\right)_{i\in\mathcal{I}}$ if, and only if, $\forall t, t' \in \mathcal{T}$ and $\forall i \in \mathcal{I}$,

1. $$\left.\begin{array}{c} \sum_{j\in\mathcal{I}\backslash\{i\}} q^*_{j,t} = \sum_{j\in\mathcal{I}\backslash\{i\}} q^*_{j,t'} \\ q^*_{i,t} \in \left[0, \overline{q}_{i,t'}\right] \\ q^*_{i,t'} \in \left[0, \overline{q}_{i,t}\right] \end{array}\right\} \Longrightarrow q^*_{i,t} = q^*_{i,t'}$$

2. $$\left.\begin{array}{c} \sum_{j\in\mathcal{I}\backslash\{i\}} q^*_{j,t} = \sum_{j\in\mathcal{I}\backslash\{i\}} q^*_{j,t'} \\ q^*_{i,t} \notin \left[0, \overline{q}_{i,t'}\right] \end{array}\right\} \Longrightarrow q^*_{i,t'} = \overline{q}_{i,t'}$$

3. $$\sum_{j\in\mathcal{I}\backslash\{i\}} q^*_{j,t} \leqslant \sum_{j\in\mathcal{I}\backslash\{i\}} q^*_{j,t'} \Longrightarrow q^*_{i,t} \in \left[\min\left\{\overline{q}_{i,t}, q^*_{i,t'}\right\}, \overline{q}_{i,t}\right]$$

[27] This is nonstandard in that I do not require that π^i have the usual form

$$\pi^i(q_i, q_{-i}) = d^{-1}\left(\sum_{j\in\mathcal{I}} q_j\right) q_i - c^i(q_i)$$

where $d : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a demand function and $c^i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a cost function.

6 Concluding remarks:

This paper studied the problem of whether or not the hypothesis that individuals behave as assumed by the concept of Nash equilibrium is falsifiable. The question is relevant since, from an epistemological point of view, the unfalsifiability of an idea casts doubts about its character of scientific knowledge. I have considered here the case of a finite number of players, each of whom is endowed with a continuous domain, perhaps subject to additional constraints. In order to increase the empirical applicability of my results, I assumed that only a finite data set is observed, and used a weak criterion of rationalizability, so as to avoid the assumption, whether implicit or explicit, that all the equilibria of the games have been collected in the finite data set. The first result that I obtain is that the hypothesis is in effect falsifiable, but that the extra assumption that equilibria are locally unique cannot be tested. The conditions that allow for falsification are suitably applied versions of the axioms of revealed preference, which must hold for each individual, conditional on the actions of his opponents. This conditionality is fairly intuitive, since only under it can one ensure that the individual is maximizing the same utility function. When players make very fine distinctions between the actions of their opponents, and when the ordinal effects of these actions on the payoff function of the individual are left unrestricted, the testable implications derived from Nash behavior are seen to be extremely mild. Nonetheless, for the general case, these implications are all the restrictions that the theory imposes, as it is shown that they are also sufficient conditions for rationalizability.

Then, the question that must be answered before concluding that, for all practical purposes, it is as if the hypothesis were unfalsifiable is whether the weak requirements for rationalizability, the finesse with which agents distinguish actions of their opponents and the arbitrariness allowed to ordinal effects are appropriate assumptions in specific cases. I have chosen to maintain weak rationalizability requirements, in order to avoid having to defend arguments that said that in the finite data set one has indeed observed all the equilibria of the games over continuous domains. To my mind, a more convenient route to explore is whether extra assumptions about how the payoffs of individuals are affected by actions of their opponents suffice to strengthen the tests of the Nash hypothesis. My findings are that, indeed, they may.

7 Appendix: Parallel Hyperplanes.

In this appendix for each $i \in \{1, ..., I\}$, $I < \infty$, e_i represents the i^{th} canonical unit vector in $\mathbb{R}^I$.

Lemma 1 *For every finite set* $\{a_t\}_{t=1}^T \subseteq \mathbb{R}^I$*, there exists* $v \in \mathbb{R}^I_{++}$ *such that for each* $t, t' \in \mathcal{T}$*,* $t \neq t'$*, it is true that* $v \cdot a_t \neq v \cdot a_{t'}$*.*

Proof. Consider the following algorithm:

Algorithm 2 *Input:* $\{a_t\}_{t=1}^T$

1. $v = \{1, ..., 1\}$, $t = 1$.
2. *If* $(\forall \tau \in \{1, ..., t-1\}) : v \cdot a_\tau \neq v \cdot a_t$, *go to 4.*
3. *Define*
$$\widehat{\tau} \in \{\tau \in \{1, ..., t-1\} | \, v \cdot a_\tau = v \cdot a_t\}$$
$$\widehat{i} = \min \{i \in \{1, ..., I\} | \, a_{i,\widehat{\tau}} \neq a_{i,t}\}$$
$$\gamma = \max \left\{ \max_{\tau \in \{1,...,t-1\}} \left\{ \left| a_{\widehat{i},\tau} - a_{\widehat{i},t} \right| \right\}, \max_{\tau,\tau' \in \{1,...,t-1\}} \left\{ \left| a_{\widehat{i},\tau} - a_{\widehat{i},\tau'} \right| \right\} \right\}$$
If $t = 2$, *define* $\varepsilon = 1$. *Else, define*
$$\varepsilon = \min \left\{ \min_{\tau \in \{1,...,t-1\} \setminus \{\widehat{\tau}\}} \{|v \cdot a_\tau - v \cdot a_t|\}, \min_{\tau,\tau' \in \{1,...,t-1\}, \tau \neq \tau'} \{|v \cdot a_\tau - v \cdot a_{\tau'}|\} \right\}$$
Define
$$\kappa = \frac{\varepsilon}{2\gamma}$$
$$v = v + \kappa e_{\widehat{i}}$$
4. *If* $t = T$, *stop. Else,* $t = t + 1$ *and go to 2.*

Output: v

Since $T < 0$, it is obvious that the algorithm runs in finite time. I now show that the v resulting at the end of the algorithm satisfies $v \in \mathbb{R}^I_{++}$ and that $\forall t, t' \in \mathcal{T}$, $t \neq t'$, it is true that $v \cdot a_t \neq v \cdot a_{t'}$.

Fix $t \in \mathcal{T}$. Suppose that before the t^{th} pass through the algorithm, $v \in \mathbb{R}^I_{++}$ is such that
$$(\forall \tau, \tau' \in \{1, ..., t-1\}, \tau \neq \tau') : v \cdot a_\tau \neq v \cdot a_{\tau'}$$
which is true when $t \leqslant 2$.

If at step (2) of the algorithm it is true that $\forall \tau \in \{1, ..., t-1\} : v \cdot a_\tau \neq v \cdot a_t$, then it is obvious from step 4 that before the $t+1^{\text{st}}$ pass through the algorithm $v \in \mathbb{R}^I_{++}$ and
$$(\forall \tau, \tau' \in \{1, ..., t\}, \tau \neq \tau') : v \cdot a_\tau \neq v \cdot a_{\tau'}$$

Now, suppose that the condition of step (2) does not hold. Then
$$(\exists \tau \in \{1, ..., t-1\}) : v \cdot a_\tau = v \cdot a_t$$
This τ is obviously unique, and is what is defined as $\widehat{\tau}$ at step (3).

Also, at step (3) it is obvious that $\varepsilon \neq 0$ if $t = 2$. Moreover for $t \geqslant 3$, the fact that $\varepsilon > 0$ follows from the uniqueness of $\widehat{\tau}$ and the assumption about v before the t^{th} pass through the algorithm. That $\widehat{i}$ is well defined follows from the fact that $\widehat{\tau} \neq t$ and the definition of set imply that $a_{\widehat{\tau}} \neq a_t$. Since $\gamma \geqslant \left| a_{\widehat{i},\widehat{\tau}} - a_{\widehat{i},t} \right| > 0$ it follows that $\kappa > 0$.

In order to avoid confusion, define $v^* = v + \kappa e_{\widehat{i}}$. Obviously, $v^* > v$, so that $v^* \in \mathbb{R}^I_{++}$. I now claim that the following three properties are satisfied by v^*:

(i)

$$v^* \cdot a_{\widehat{\tau}} \neq v^* \cdot a_t$$

(ii)

$$(\forall \tau \in \{1, ..., t-1\} \setminus \{\widehat{\tau}\}) : v^* \cdot a_\tau \neq v^* \cdot a_t$$

(iii)

$$(\forall \tau, \tau' \in \{1, ..., t-1\}, \tau \neq \tau') : v^* \cdot a_\tau \neq v^* \cdot a_{\tau'}$$

For the claim (i), just notice that

$$\begin{aligned} v^* \cdot a_{\widehat{\tau}} - v^* \cdot a_t &= v \cdot a_{\widehat{\tau}} + \kappa a_{\widehat{i},\widehat{\tau}} - v \cdot a_t - \kappa a_{\widehat{i},t} \\ &= \kappa \left(a_{\widehat{i},\widehat{\tau}} - a_{\widehat{i},t} \right) \\ &\neq 0 \end{aligned}$$

since $v \cdot a_{\widehat{\tau}} = v \cdot a_t$, $\kappa > 0$ and $a_{\widehat{i},\widehat{\tau}} - a_{\widehat{i},t} \neq 0$.

If $t = 2$, claims (ii) and (iii) are trivial. Hence I now assume that $t \geqslant 3$.

Consider first claim (ii). Fix $\tau \in \{1, ..., t-1\} \setminus \{\widehat{\tau}\}$. By construction $v \cdot a_\tau \neq v \cdot a_t$. Suppose first that $v \cdot a_\tau < v \cdot a_t$. Then

$$\begin{aligned} v^* \cdot a_\tau - v^* \cdot a_t &= v \cdot a_\tau + \kappa a_{\widehat{i},\tau} - v \cdot a_t - \kappa a_{\widehat{i},t} \\ &= v \cdot a_\tau - v \cdot a_t + \kappa \left(a_{\widehat{i},\tau} - a_{\widehat{i},t} \right) \\ &\leqslant v \cdot a_\tau - v \cdot a_t + \kappa \left| a_{\widehat{i},\tau} - a_{\widehat{i},t} \right| \\ &\leqslant v \cdot a_\tau - v \cdot a_t + \kappa \gamma \\ &= v \cdot a_\tau - v \cdot a_t + \frac{\varepsilon}{2} \\ &\leq -\varepsilon + \frac{\varepsilon}{2} \\ &= -\frac{\varepsilon}{2} \\ &< 0 \end{aligned}$$

where the sixth step follows since $v \cdot a_\tau - v \cdot a_t = -|v \cdot a_\tau - v \cdot a_t|$ and $|v \cdot a_\tau - v \cdot a_t| \geqslant \varepsilon$.

If, on the other hand, $v \cdot a_t < v \cdot a_\tau$, then

$$\begin{aligned}
v^* \cdot a_t - v^* \cdot a_\tau &= v \cdot a_t + \kappa a_{\widehat{i},t} - v \cdot a_\tau - \kappa a_{\widehat{i},\tau} \\
&= v \cdot a_t - v \cdot a_\tau + \kappa \left(a_{\widehat{i},t} - a_{\widehat{i},\tau} \right) \\
&\leqslant v \cdot a_t - v \cdot a_\tau + \kappa \left| a_{\widehat{i},t} - a_{\widehat{i},\tau} \right| \\
&\leqslant v \cdot a_t - v \cdot a_\tau + \kappa \gamma \\
&= v \cdot a_t - v \cdot a_\tau + \frac{\varepsilon}{2} \\
&\leq -\varepsilon + \frac{\varepsilon}{2} \\
&= -\frac{\varepsilon}{2} \\
&< 0
\end{aligned}$$

where the sixth step follows since $v \cdot a_t - v \cdot a_\tau = -|v \cdot a_t - v \cdot a_\tau|$ and $|v \cdot a_t - v \cdot a_\tau| \geqslant \varepsilon$.

For claim (iii), fix $\tau, \tau' \in \{1, ..., t-1\}$, $\tau \neq \tau'$. By assumption about v before the t^{th} pass through the algorithm, $v \cdot a_\tau \neq v \cdot a_{\tau'}$, so that we can assume without loss of generality that $v \cdot a_\tau < v \cdot a_{\tau'}$. As before,

$$\begin{aligned}
v^* \cdot a_\tau - v^* \cdot a_{\tau'} &= v \cdot a_\tau + \kappa a_{\widehat{i},\tau} - v \cdot a_{\tau'} - \kappa a_{\widehat{i},\tau'} \\
&= v \cdot a_\tau - v \cdot a_{\tau'} + \kappa \left(a_{\widehat{i},\tau} - a_{\widehat{i},\tau'} \right) \\
&\leqslant v \cdot a_\tau - v \cdot a_{\tau'} + \kappa \left| a_{\widehat{i},\tau} - a_{\widehat{i},\tau'} \right| \\
&\leqslant v \cdot a_\tau - v \cdot a_{\tau'} + \kappa \gamma \\
&= v \cdot a_\tau - v \cdot a_{\tau'} + \frac{\varepsilon}{2} \\
&\leq -\varepsilon + \frac{\varepsilon}{2} \\
&= -\frac{\varepsilon}{2} \\
&< 0
\end{aligned}$$

where the sixth step follows since $v \cdot a_\tau - v \cdot a_{\tau'} = -|v \cdot a_\tau - v \cdot a_{\tau'}|$ and $|v \cdot a_\tau - v \cdot a_{\tau'}| \geqslant \varepsilon$.

This claims show that before the $t+1^{\text{st}}$ pass through the algorithm

$$(\forall \tau, \tau' \in \{1, ..., t\}, \tau \neq \tau') : v \cdot a_\tau \neq v \cdot a_{\tau'}$$

■

8 Appendix: Construction of strongly concave preferences.

Let $I \in \mathbb{N}$. For a finite sequence $(a_t, v_t)_{t=1}^{T}$, where $(\forall t \in \mathcal{T}) : a_t \in \mathbb{R}_+^I$ and $v_t \in \mathbb{R}_{++}^I$, define the following two conditions:

Condition 2 $\forall N \leqslant T$ *and* $\forall \{\tau_1, ..., \tau_N\} \subseteq \mathcal{T}$,

$$\left((\forall n \in \{1, ..., N-1\}) : v_{\tau_n} \cdot \left(a_{\tau_{n+1}} - a_{\tau_n}\right) \leqslant 0\right) \Longrightarrow \left\{ \begin{array}{c} a_{\tau_1} = a_{\tau_N} \\ or \\ v_{\tau_N} \cdot (a_{\tau_1} - a_{\tau_N}) > 0 \end{array} \right.$$

Condition 3 *Besides condition 2,*

$$(\forall t, t' \in \mathcal{T}) : v_t \neq v_{t'} \Longrightarrow a_t \neq a_{t'}$$

The following theorem is derived from a result in Chiappori and Rochet (1987), which is obtained in a different context, but has the same mathematical content.

Theorem 6 *If* $(a_t, v_t)_{t=1}^{T}$ *be a finite sequence, where* $(\forall t \in \mathcal{T}) : a_t \in \mathbb{R}_+^I$ *and* $v_t \in \mathbb{R}_{++}^I$*, satisfies condition 3, then* $\exists V : \prod_{i \in \mathcal{I}} \left[0, \overline{\overline{a}}_i\right] \longrightarrow \mathbb{R}$*, differentiable and strongly concave, such that*

$$(\forall t \in \mathcal{T}) \left(\forall a \in \prod_{i \in \mathcal{I}} A^i\right) : V(a) > V(a_t) \Longrightarrow v_t \cdot a > v_t \cdot a_t$$

and

$$(\forall t \in \mathcal{T}) \left(\forall a \in \prod_{i \in \mathcal{I}} A^i\right) : v_t \cdot a < v_t \cdot a_t \Longrightarrow V(a) < V(a_t)$$

Proof. Since $\prod_{i \in \mathcal{I}} \left[0, \overline{\overline{a}}_i\right]$ is compact, it follows from condition 3 and the theorem in Chiappori and Rochet (1987),[28] that there exists $V : \prod_{i \in \mathcal{I}} \left[0, \overline{\overline{a}}_i\right] \longrightarrow \mathbb{R}$, infinitely differentiable and strongly concave, such that

$$(\forall t \in \mathcal{T}) : a_t \in Arg \max_{a \in R_+^I : v_t \cdot a \leqslant v_t \cdot a_t} V(a)$$

Now, fix $t \in \mathcal{T}$ and $a \in \prod\limits_{i \in \mathcal{I}} A^i$. If $V(a) > V(a_t)$, it follows directly from the previous condition that $v_t \cdot a > v_t \cdot a_t$. If, on the other hand, $v_t \cdot a < v_t \cdot a_t$ and $V(a) \geqslant V(a_t)$, it then follows that $\forall \theta \in (0, 1)$

$$v_t \cdot (\theta a + (1 - \theta) a_t) < v_t \cdot a_t$$

whereas, by strong concavity,

$$V(\theta a + (1 - \theta) a_t) > V(a_t)$$

which contradicts the previous condition. ■

[28] Displacing the origin of $\mathbb{R}_+^I$ if necessary.

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