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Montecarlo simulation of long-term dependent processes: a primer [†]

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Abstract

As a natural extension to León and Vivas (2010) and León and Reveiz (2010) this paper briefly describes the Cholesky method for simulating Geometric Brownian Motion processes with long-term dependence, also referred as Fractional Geometric Brownian Motion (FBM).

Results show that this method generates random numbers capable of replicating independent, persistent or antipersistent time-series depending on the value of the chosen Hurst exponent. Simulating FBM via the Cholesky method is (i) convenient since it grants the ability to replicate intense and enduring returns, which allows for reproducing well-documented financial returns' slow convergence in distribution to a Gaussian law, and (ii) straightforward since it takes advantage of the Gaussian distribution ability to express a broad type of stochastic processes by changing how volatility behaves with respect to the time horizon. However, Cholesky method is computationally demanding, which may be its main drawback.

Potential applications of FBM simulation include market, credit and liquidity risk models, option valuation techniques, portfolio optimization models and payments systems dynamics. All can benefit from the availability of a stochastic process that provides the ability to explicitly model how volatility behaves with respect to the time horizon in order to simulate severe and sustained price and quantity changes. These applications are more pertinent than ever because of the consensus regarding the limitations of customary models for valuation, risk and asset allocation after the most recent episode of global financial crisis.

Keywords: Montecarlo simulation, Fractional Brownian Motion, Hurst exponent, Long-term Dependence, Biased Random Walk.

JEL Classification: C15, C53, C63, G17, G14.

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1. Introduction

As a natural extension to León and Vivas (2010) and León and Reveiz (2010) this paper briefly describes the Cholesky method for simulating Geometric Brownian Motion processes with long-term dependence, also referred as Fractional Geometric Brownian Motion. This choice results from its parsimony, simplicity and documented theoretical advantages (Jennane *et al.*, 2001).

Price or return paths simulated with this method exhibit approximately the same first four distributional moments for differing cases of serial-dependence (*i.e.* independence, persistence and antipersistence). Graphical inspection (*i.e.* probability plots and simulated paths) and the estimation of the simulated time-series' Hurst exponent corroborates the impact of long-term memory in financial markets, where significant and sustained price changes are more likely than standard Brownian Motion or random walk models assume.

Results show that the chosen method generates random numbers capable of replicating independent, persistent or antipersistent time-series depending on the value of the chosen Hurst exponent. Simulating FBM via the Cholesky method is (i) convenient since it grants the ability to replicate intense and enduring returns, which allows for reproducing well-documented financial returns' slow convergence in distribution to a Gaussian law, and (ii) straightforward since it takes advantage of the Gaussian distribution ability to express a broad type of stochastic processes by changing how volatility behaves with respect to the time horizon. A drawback is that the methodology is computationally demanding, principally from the use of the Cholesky decomposition.

Potential applications of Fractional Geometric Brownian Motion simulation include market, credit and liquidity risk models, option valuation techniques, portfolio optimization models and payments systems dynamics. All can benefit from the availability of a stochastic process that provides the ability to explicitly model how volatility behaves with respect to the time horizon in order to simulate severe and sustained price and quantity changes. These applications are more pertinent than ever because of the consensus regarding the limitations of customary models for valuation, risk and asset allocation after the most recent episode of global financial crisis.

2. Fractional Brownian Motion (FBM)

Bachelier (1900) introduced Brownian motion for describing the behavior of financial prices in what is known as Arithmetic Brownian motion, which was afterwards revised by Samuelson (1965) because of several inconveniences arising from applying such process to financial prices directly (*e.g.* prices may turn negative in the long-term); Samuelson´s revision resulted in applying Arithmetic Brownian motion to prices' log-returns, commonly referred as Geometric Brownian Motion.

The Geometric Brownian Motion (henceforth Brownian Motion or BM) underlies modern economic and financial theory. Following Mandelbrot (1963), if *Z(t)* is the log-return of the price of a stock at the end of time period *t*, successive differences of the form *Z(t+s) – Z(t)* are (i) independent, (ii) Gaussian or normally distributed, (iv) random variables (iv) with zero mean and (v) variance proportional to the differencing interval *s*. Therefore, let σ^2 be the variance and *Z(•)* an independent and continuous process, BM may be expressed as in [F1], where the operator ≜ means independently and equally distributed:

$$Z(t+s) - Z(t) \triangleq \mathcal{N}(0, \sigma^2) \quad \text{[F1]}$$

Consequently, the resulting process of *Z(t)* exhibits the following statistical properties:

$$E[Z(t)] = 0, \forall t \quad \text{[F2]}$$

$$COV[Z(t,s)] = 0;\ \forall t, \forall s \quad \text{[F3]}$$

These two properties of the BM result in the widespread weak-form of the Efficient Market Hypothesis (EMH): [F2] states that the expected return is zero for any time-horizon, thus the current price is the best forecast for future price, whereas [F2] affirms that past behavior of returns is irrelevant.

Independence is the foremost important pillar of BM, even more vital than the Gaussian distribution assumption. Two main reasons support this statement: First, even if returns are not normally distributed, the Central Limit Theorem demonstrates that independent time-series of returns will converge in distribution to a Gaussian Law rather quickly. Moreover, even if time-series exhibit some sort of short-term dependence which fades out after some realizations of the random variable (*e.g.* the process exhibits weak-dependence, such as AR or GARCH effects), Central Limit Theorem also guarantees such convergence.

Second, if time-series are independent or weak-dependent it is also true that variance of the process is proportional to the differencing interval; this is, that the square of the fluctuations of the prices increase in proportion to the time scale. According to Sornette (2003) this is equivalent to saying that the typical amplitude of returns is proportional to the square root of the time scale, which he describes as the most important prediction of the BM model. This is the widespread square-root-of-time-rule (hereafter referred as SRTR), which simply consists of multiplying the standard deviation calculated from high-frequency (*hf*) time-series (*e.g.* daily) by the square-root of n, where n is the number of units which compose the low-frequency (*lf*) time-series (*e.g.* yearly), as in [F4].

$$\sigma_{lf} = \sigma_{hf} n^{0.5};\ \forall n \quad \text{[F4]}$$

As acknowledged by Malavergne and Sornette (2006), slow convergence in distribution of financial time-series to a Gaussian law – even for low frequency returns - may be caused by significant time-dependencies between asset returns. Therefore, testing for independence of financial time-series is as important as the more traditional focus on non-normality of returns.

Following Hurst (1951) work on long-term dependence in Geophysics, seminal work by Mandelbrot (1972) demonstrated the presence of long-term memory in financial assets' time-series. Peters (1992) acknowledged Mandelbrot´s contribution and discarded the ability of prevalent short-term memory econometric models (*e.g.* AR, ARMA, GARCH) to capture or replicate the type of enduring dependence found in financial time-series.

Due to the evidence of long-term dependence in financial returns, which has been confirmed by Nawrocki (1995), Peters (1996), Willinger *et al.* (1999), Weron and Przybylowicz (2000), Sun *et al.* (2007), Menkens (2007), Cajueiro and Tabak (2008), León and Vivas (2010) and León and Reveiz (2010), the SRTR is inappropriate to describe the way financial returns scale with time. Hence, according to Jennane *et al.* (2001) and McLeod and Hipel (1978), [F3] and [F4] may be generalized as in [F5] and [F6], respectively, where *H* is known as the Hurst exponent[1] and m corresponds to the resolution or frequency of the time-series (*e.g.*, if daily autocovariance is to be estimated, then $m = 1$):

$$COV_m[Z(t,s)] = \frac{\sigma_m^2}{2}\left[|t-s+m|^{2H} - 2|t-s|^{2H} + |t-s-m|^{2H}\right];\ \forall t, \forall s \qquad \text{[F5]}$$

$$\sigma_{lf} = \sigma_{hf} n^H; \forall n \qquad \text{[F6]}$$

As stated by Sun *et al.* (2007), in the *H*=0.5 and *H*≈0.5 cases the process has no memory –is independent-, hence next period's expected result has the same probability of being lower or higher than the current result, and the autocovariance or autocorrelation resulting from [F5] is zero[2] (Jennane *et al.*, 2001). Applied to financial time-series this is analogous to assuming that the process followed by assets' returns is similar to coin tossing, where the probability of heads (*e.g.* rise in the price) or tails (*e.g.* fall in the price) is the same (½), and is independent of every other toss; this is precisely the theoretical base of the Capital Asset Pricing Model (CAPM), the Arbitrage Pricing Theory (APT), the Black & Scholes model and the Modern Portfolio Theory (MPT).

When *H* takes values between 0.5 and 1 (0.5<*H*≤1) evidence suggests a persistent behavior, therefore one must expect the result in the next period to be similar to the current one (Sun *et al.*,

[1] Hurst exponent (*H*) is named after British physicist H.E. Hurst (1880–1978), whose analysis suggested that numerous natural phenomena significantly diverged from being long-term independent as assumed by Geophysics at that time. Hurst suggested ignoring the customary SRTR and developed a methodology for estimating the empirical exponent which fits how the random variable behaves with respect to time. A revised version of Hurst's methodology was developed by Mandelbrot and Wallis (1969) under the name of Rescaled Range Analysis (R/S); in this document an adjusted version of Mandelbrot and Wallis' *H* is used (León and Reveiz, 2010; León and Vivas, 2010). The unfamiliar reader may refer to the previously mentioned documents and Peters (1989, 1992, 1994, 1996).

[2] Please note that if *H=0.5* in [F5] FBM reduces to BM; this is, the right side of the equation equals cero.

2007), and the autocovariance or autocorrelation resulting from [F5] is positive (Jennane *et al.*, 2001). According to Menkens (2007) this means that increments are positively correlated: if an increment is positive, succeeding increments are most likely to be positive than negative. In other words, each event has influence on future events; therefore there is dependence or memory in the process.

As *H* becomes closer to one (1) the range of possible future values of the variable will be wider than the range of purely random and independent variables. Peters (1996) argues that the presence of persistence is a signal that today's behavior doesn't influence near future only, but distant future as well.[3]

On the other hand, when *H* takes values below 0.5 ($0 \leq H < 0.5$) there is a signal that suggests an antipersistent behavior of the variable. This means, as said by Sun *et al.* (2007), that a positive (negative) return is more likely followed by negative (positive) ones, and the autocovariance or autocorrelation resulting from [F5] is negative (Jennane *et al.*, 2001); hence, as stated by Mandelbrot and Wallis (1969), this behavior causes the values of the variable to tend to compensate with each other, avoiding time-series' overshooting. Applied to financial markets series, Menkens (2007) affirms that this kind of continuously compensating behavior would suggest a constant overreaction of the market, one that would drive it to a permanent adjustment process. Similarly, Peters (1996) links this behavior to the well-known "mean-reversion" process.

Mandelbrot's research not only applied a revised version of Hurst's developments, but introduced a novel process to describe the behavior of financial prices, where the idealistic smoothness and continuity of BM is conveniently harmonized with the factual roughness and discontinuity of financial markets. This generalization of BM grants the process the ability to scale with different powers of time while preserving the shape of the distribution of returns (*e.g.* returns are scale-invariant), which indicates that returns on different time-scales can be re-scaled from or to the original process without modifying its statistical properties. Such generalization was baptized as Fractional Brownian Motion (FBM) because it implies that the process is self-similar or self-affine, which means that fractions or parts of the process preserve the characteristics and are related to the whole or other *fractions* of the process; therefore the term "fractal" for Mandelbrot's theoretical developments.

Consequently, as defined by Greene and Fielitz (1979), FBM is a self-similar process which follows a n^H law, where its increments (*e.g. fractional Gaussian noise*) exhibit long-term dependence, with the Hurst exponent (*H*) determining the direction and intensity of such dependence. One of the key characteristics of FBM is that the span or window of interdependence can be said to be

[3] Some explanations for financial assets' return persistence are found in human behavior, since the latter contradicts rationality assumption in several ways (*e.g.* investors' choices are not independent, and they are characterized by non-linear and imitative behavior; investors resist to change their perception until a new credible trend is established, and investors don't react to new information in a continuous manner, but rather in a discrete and cumulative way). Other explanations for financial assets' return persistence have to do with the importance of economic fundamentals, and the use of privileged information. Alternatively, some authors, conclude that markets' liquidity make instantaneous trading impossible, leading to transactions' splitting and decisions' clustering, resulting in market prices which don't fully reflect information immediately, but incrementally. León and Reveiz (2010) contains references for each explanation.

infinite, contrasting traditional Markov processes (Mandelbrot and Van Ness, 1968) and making any application of the Central Limit Theorem (*e.g.* to assume normality) flawed at best.

3. Montecarlo simulation of *fractional Gaussian noise* and FBM

Customary Montecarlo simulation of a *Gaussian noise* (*e.g. white noise*) consists of the simulation of a considerably large number of independent and normally distributed random numbers with zero mean and unit variance, and the application of Euler's discrete-time approximation for a BM process. Let the process' mean and standard deviation be denoted by scalars μ and σ, respectively; a column vector of p-random numbers be denoted by ε, where $\varepsilon \sim \mathcal{N}(0,1)$; Y_t the asset's log-return during time *t*; X_t the asset's price at the end of time *t*, then [F7] and [F8] exhibit next period's $(t+1)$ vectors of p-simulated log-returns (Y_{t+1}) and of p-prices (X_{t+1}), respectively, which correspond to the mentioned *Gaussian* noise and BM:

$$Y_{t+1} = \mu + \sigma\varepsilon \; ; \; \varepsilon \sim \mathcal{N}(0,1) \qquad \text{[F7]}$$

$$X_{t+1} = X_t e^{Y_{t+1}} \qquad \text{[F8]}$$

When simulating more than one-period-in-the-future prices the p-sized ε column vector of independent and normally distributed random numbers becomes a $p \times q$ matrix, where p still corresponds to the number of simulations, whereas q corresponds to the number of periods-in-the-future to be simulated; please note that since the random numbers are generated independently there is no time-dependence. Because log-returns are used to estimate the moments of the process, returns may be accumulated through n periods as in [F9], which results in $Y_{t \to t+n}$. After calculating $Y_{t \to t+n}$ it is straightforward to obtain a $p \times q$ matrix $X_{t \to t+n}$ which contains the successive prices of the variable as in [F10].

$$Y_{t \to t+n} = \sum_{u=1}^{n} (Y_{t+u}) \qquad \text{[F9]}$$

$$X_{t \to t+n} = X_t e^{Y_{t \to t+n}} \qquad \text{[F10]}$$

It is usual to employ the simulation procedure just described to simulate *Gaussian noise* for more than one asset without disregarding the dependence existing across assets' returns. This is customarily carried out by generating a $p \times a \times q$ hyper-matrix of independent and normally distributed random numbers with zero mean and unit variance, where p and q stand for the number of simulations and the number of periods-in-the-future to be simulated, whereas a corresponds to the number of assets to be considered. Each q-layer of the hyper-matrix is afterwards multiplied by the triangular matrix resulting from applying the Cholesky decomposition to the assets' returns correlation matrix[4], which at the end yields q-matrixes of normally distributed random numbers with zero mean and variance equal to the covariance matrix of assets' returns; such hyper-matrix is denoted herein as $\ddot{\varepsilon}$.

Let Σ and Θ stand for the estimated correlation matrix and for the Cholesky decomposition procedure, respectively, the first-period-in-the-future $(t+1)$ case of cross-dependency for each a-asset is found in the first layer of the hyper-matrix $(\ddot{\varepsilon}_1)$, as in [F12]. Subsequently, in order to simulate the a-asset first-period-in-the-future $(t+1)$ returns and prices a-asset's mean (μ_a) and standard deviation (σ_a), along with the a-column from $(\ddot{\varepsilon}_1)$ are used as in [F13] and [F14].

$$\ddot{\varepsilon}_1 = \varepsilon_1 \times \Theta(\Sigma) \qquad \text{[F12]}$$

$$Y_{a,t+1} = \mu_a + \sigma_a \ddot{\varepsilon}_{a,1} \; ; \; \ddot{\varepsilon}_{a,1} \sim \mathcal{N}(0,\Sigma) \qquad \text{[F13]}$$

$$X_{a,t+1} = X_{a,t} e^{Y_{a,t+1}} \qquad \text{[F14]}$$

Similar to the procedure previously presented, which transforms independent and normally distributed random numbers (ε) into cross-dependent and normally distributed numbers $(\ddot{\varepsilon})$, it is also possible to simulate a *fractional Gaussian noise*. The major change consists of switching from asset-dependence to serial or time-dependence, which entails estimating autocovariance instead of covariance.

Despite it is tempting to use standard autocovariance estimation, as acknowledged by Hurst (1951), Mandelbrot (1972) and Peters (1994), its consistency and robustness for detecting and measuring dependence, especially long-term dependence, is critically restricted to the Gaussian world; this is, using customary autocovariance o autocorrelation is non-robust to changes in distribution, and should be used cautiously if time-series are not normal. However, as exhibited in [F5], an alternative method for assessing time-dependence is available, where the degree and sign of the dependence (*e.g.* persistence or antipersitence) is determined jointly by the variance, the Hurst exponent (*H*) and the n-period-in-the-future to be simulated.

Let Ψ and Θ be the autocorrelation matrix resulting from the autocovariance estimated as in [F5] and the Cholesky decomposition procedure, correspondingly. Transforming a $1 \times q$ vector of independent and normally distributed random numbers (ε) into time-dependent and normally distributed numbers $(\bar{\bar{\varepsilon}})$ is as in [F15], and $X_{t \to t+n}$ in [F17] exhibits a n-periods-in-the-future simulated single path. Simulating p-price paths becomes straightforward if ε is a $p \times n$ matrix.

[4] The Cholesky decomposition of the correlation matrix (Σ) consists of estimating matrix L in [F11], where L is a positive-defined lower-triangular-matrix. If Σ is a g-size square correlation matrix, multiplying the resulting g-size square matrix L to a $n \times g$ matrix of uncorrelated samples produces a $n \times g$ matrix of correlated samples, where their correlation approximates Σ.

$$\Sigma = LL^T \qquad \text{[F11]}$$

The reader is referred to Cuthberson and Nitzsche (2004) for a more comprehensive explanation.

$$\bar{\bar{\varepsilon}} = \varepsilon \times \Theta(\Psi) \quad \text{[F15]}$$

$$Y_{t \to t+n} = \mu + \sigma \bar{\bar{\varepsilon}}\,;\, \bar{\bar{\varepsilon}} \sim \mathcal{N}(0, \Psi) \quad \text{[F16]}$$

$$X_{t \to t+n} = X_t e^{Y_{t \to t+n}} \quad \text{[F17]}$$

4. Montecarlo simulated FBM

This section displays some illustrative results based on the procedure previously exhibited. The first two moments of the distribution (*i.e.* mean and standard deviation) will be 0.00 and 0.01, and will remain constant across the different cases of Hurst[5] exponents herein considered. Dependence will be estimated as in [F5], simulations as in [F17], and three cases of Hurst exponents (H) will be considered (Table 1).

Table 1
Three cases of Hurst exponents[6]

Case	Description
$H_{ant} = 0.4$	Corresponds to long-term antipersistence
$H_{ind} = 0.5$	Corresponds to long-term non-dependence
$H_{per} = 0.6$	Corresponds to long-term persistence

Source: authors' design

Simulating five price paths for 500-periods-in-the-future with the three cases of Hurst exponents results in the following plots (Figure 1). Despite being a small number of simulations, it is clear that each case has unique characteristics related to the way that time-series diffuse across time: antipersitent (persistent) simulated time-series exhibit the lowest (highest) dispersion around the mean. Note that the starting price (100) and random numbers used to simulate the processes (ε) are the same for the three cases of Hurst exponents; therefore, differences between cases are only due to the transformation of random variables when multiplied by the corresponding

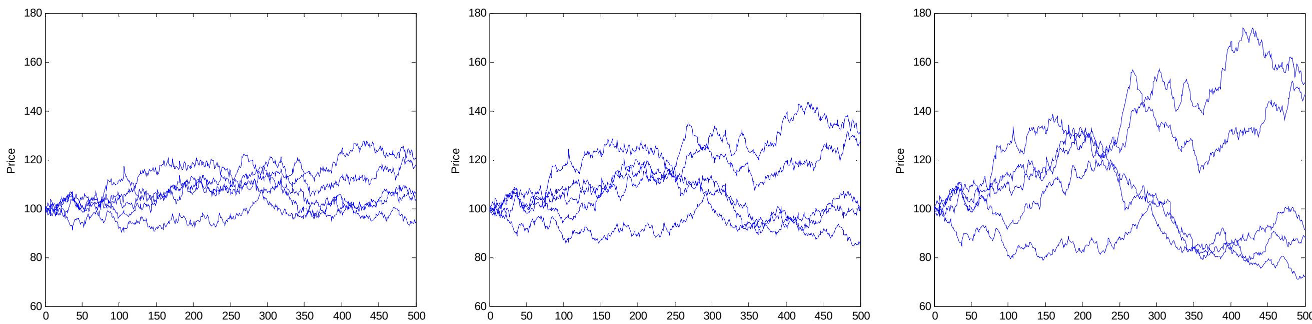
[5] Please note that the adjusted Hurst exponent developed by León and Vivas (2010) and León and Reveiz (2010) should be used. This adjustment is convenient since non-adjusted Hurst exponent is estimated under the assumption of infinite or near-infinite-length time-series, which yields biased –overestimated- Hurst exponents when using finite-time-series; thus, not adjusting the Hurst exponent for such bias yields unreliable –persistent- results.

[6] Please note that these cases correspond to factual figures. Based on results by León and Reveiz (2010) and León and Vivas (2010), independent or near independent processes (H=0.5 and H≈0.5) were found for S&P500, MSCI World and S&P Agriculture & Live Stock indexes; antipersistence was found for S&P Precious Metals (0.48) and US energy indexes (between 0.34 and 0.42), where only the latter were significant at 95% confidence level; persistence was found for Colombian stock and fixed income indexes (*i.e.* IGBC and IDXTES, 0.61 and 0.60, respectively), MSCI Emerging markets index (0.59) and EMBI price index (0.59), all significant at 95% confidence level.

Cholesky decomposition of the covariance matrix as in [F15]. This will allow for objective comparisons between the three cases.

Figure 1
Three cases of Hurst exponents: 500 daily returns (5 simulations)

$H_{ant} = 0.4$ $H_{ind} = 0.5$ $H_{per} = 0.6$



Source: authors' calculations

It is remarkable to find that despite the graphical difference between the simulated time-series is rather clear, their distributions' first, second, third and fourth moments do not differ greatly from each other; this is, as exhibited in Table 2, they asymptotically preserve the mean (0.00), standard deviation (0.01), skewness (0.00) and kurtosis (3.00). These results converge with Sornette (2003) regarding the inability of frequency distributions to capture other types of structures (*i.e.* serial-dependence) typical of complex time-series such as financial returns.

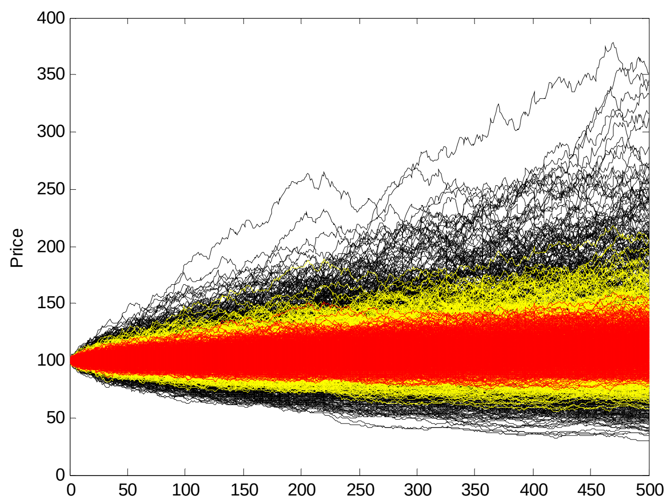
Table 2
Three cases of Hurst exponents: 500 daily return's statistical properties (5 simulations)

Case	Simulation	Mean	Std. Dev.	Skewness	Kurtosis
$H_{ant} = 0.4$	1	0.0000	0.0097	0.1317	3.1336
	2	-0.0000	0.0104	0.0855	2.6112
	3	0.0004	0.0101	-0.1388	2.8518
	4	0.0002	0.0094	-0.0271	3.1076
	5	0.0005	0.0101	0.0083	3.2630
$H_{ind} = 0.5$	1	-0.0000	0.0097	0.0885	3.1151
	2	-0.0002	0.0103	0.0768	2.6063
	3	0.0006	0.0101	-0.1431	2.8284
	4	0.0002	0.0093	-0.0575	3.1332
	5	0.0009	0.0101	0.0277	3.1620
$H_{per} = 0.6$	1	-0.0002	0.0096	0.0449	3.0433
	2	-0.0005	0.0101	0.0588	2.6415
	3	0.0010	0.0101	-0.1422	2.7798
	4	0.0002	0.0092	-0.0803	3.1625
	5	0.0016	0.0100	0.0548	3.0622
Assumed		**0.0000**	**0.0100**	**0.0000**	**3.0000**

Source: authors' calculations

In order to further assess the difference between the three dependence cases, 5,000 price paths for each Hurst exponent case were simulated (Figure 2). The difference in the way each case simulates prices is unmistakable: the wider paths (black) correspond to the persistence case, the narrower (red) to the antipersistence case, and the intermediate (yellow) to the independence case. As expected, the higher the Hurst exponent, the broader the range of simulated prices.

Figure 2
5,000 price paths for each case of Hurst exponent



Source: authors' calculations

As exhibited in Table 3, the mean of the first four distributional moments throughout the 5,000 simulated paths show that they do not differ significantly across the three cases, whilst the mean adjusted Hurst exponent effectively replicates each dependence case. Once again, this confirms frequency distributions and their moments failing to capture other type of structures in time-series, such as serial dependence. This is rather relevant since it is well-documented that focusing on the frequency of returns ignores the importance of their sequence, which could be the main reason why Value at Risk, Expected Shortfall or Extreme Value Theory tend to underestimate risk (Malevergne and Sornette, 2006; Los, 2005; Sornettte, 2003), whilst customary use of GARCH models is insufficient to account for observed volatility.[7]

[7] Sornette (2003) demonstrates that the financial industry's standard GARCH (1,1), even when using a t-distribution with 4 degrees of freedom, is unable to account for the dependencies observed in real DJIA data. In order to account for such dependence Sornette (2003) and Reveiz and León (2010) suggest using a measure such as drawdowns.

Table 3
Three cases of Hurst exponents: 2,500 daily return's statistical properties [a]
(Mean of 5,000 simulations)

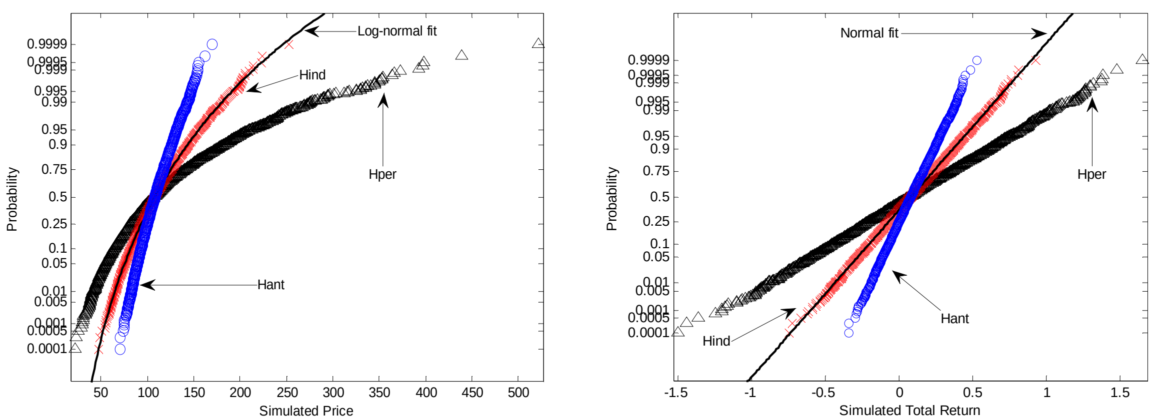
Case	Mean	Std. Dev.	Skewness	Kurtosis	Adj. H Exp.
$H_{ant} = 0.4$	0.0002	0.0099	0.0012	2.9992	0.4199
$H_{ind} = 0.5$	0.0002	0.0099	0.0012	2.9998	0.5020
$H_{per} = 0.6$	0.0002	0.0098	0.0011	2.9992	0.5816
Assumed	**0.0000**	**0.0100**	**0.0000**	**3.0000**	

[a] Estimations are based on a 2,500 daily returns time-series; this is due to documented issues regarding the inconvenience of using short series for estimating this the adjusted Hurst exponent (León and Vivas, 2010; León and Reveiz, 2010).
Source: authors' calculations.

Instead of focusing on the distribution of each simulation it is advisable to focus on the end-of-the-simulations' prices and the total returns. Figure 3 exhibits the probability plots for the end-of-the-simulations' prices (left panel) and for the total returns (right panel), for each Hurst exponent case.

Figure 3
Probability plots

End-of-the-simulations' prices | Total returns



Source: authors' calculations

The independence case corresponds to the normality-of-returns assumption (*i.e.* log-normality-of-prices) with original (assumed) mean and standard deviation. With the same volatility input the persistence case exhibits a significant departure from the customary BM, where the most evident characteristic is the higher frequency and the larger magnitude of extreme returns when compared to the independence case; the converse is also true, where antipersistence results in the lower frequency and the smaller magnitude of extreme returns.

It is remarkable that Figure 3 also displays that persistence and antipersistence cases approximately preserve the independent's case shape. For instance, the right panel of Figure 3 depicts that the three cases' simulated returns may be approximated through a linear fit, which corresponds to the Gaussian or normality assumption. Each case displays a particular slope, where the higher the dependence the lower the slope; this is, the more persistent (antipersistent) the time-series, the larger and more likely (the smaller and less likely) the simulated returns when compared with customary BM under the independence assumption.

Such interesting results regarding Figure 3 correspond to an attribute typical of the Gaussian distribution highlighted by Haug and Taleb (2009): it is possible to express any probability distribution in terms of Gaussian, even if it has fat tails, by varying the standard deviation at the level of the density of the random variable. In the case in hand, the Gaussian can express a broad type of stochastic processes by changing how volatility behaves with respect to the time horizon.

5. Final remarks

BM is insufficient to describe some well-known characteristics of financial time-series, where short-term volatility clustering is the most documented and modeled. Nevertheless, as documented by León and Vivas (2010) and León and Reveiz (2010), long-term dependence has been largely ignored despite its presence on financial time-series has been widely verified by several authors (*e.g.* Mandelbrot, 1972; Los, 2003; Peters, 2004; Menkens, 2007).

An appealing alternative for tackling some of the flaws of BM is to generalize it in order to recognize the presence of long-term memory. Such alternative consists of using factual data for estimating the way volatility scales throughout time. Fractal theory, developed by Mandelbrot, comprises some techniques to reach that alternative, including what is commonly referred as FBM.

As demonstrated in this paper, combining FBM and Hurst exponents typical of financial time-series allows for simulating processes which display characteristics analogous to those found in financial markets, where serial dependence may result in acute and sustained price changes throughout time. With the same expected (mean) return and dispersion (standard deviation) FBM is capable of getting closer to the true behavior of financial time-series, where large and continuous returns are more likely than what standard BM assumes or is able to simulate. This results from the Gaussian distribution's ability to express a broad type of stochastic processes by changing how volatility behaves with respect to the time horizon.

Simulating prices or returns that comply with an estimated persistence metric such as the Hurst exponent is useful. Besides recognizing the impact of investment's horizon for portfolio optimization (León and Reveiz, 2010), market, credit and liquidity risk assessment may profit from the ability to model –mostly overlooked- large and continuous prices' and quantities' changes. The most recent episode of financial crisis provides two relevant insights: (i) according to the IMF (2010), although liquidity risk management tools existed prior to the most recent episode of financial crisis, they were unprepared for a large and long-lasting shock; (ii) as a consequence of the failure of volatility scaling methods, the Basel Committee on Banking Supervision (BIS, 2009) changed the quantitative standards for calculating Value at Risk, where the customary use of the SRTR has been severely restricted to those cases in which technical support exists.

Other applications of FBM simulation are also straightforward. Processes exhibiting significant dependence may profit from its convenience. A topic worth approximating with this approach is payments' simulation for identifying and assessing sources of systemic risk within a payments system. Instead of using historical simulation methods for simulating payments within the payments system as in León *et al*. (2011), being able to simulate payments without ignoring their documented persistence (Bouchaud *et al.*, 2008; Lillo and Farmer, 2004) allows for more flexible modeling.

Nevertheless, some challenges still remain. First, as documented by Di Mateo (2007), the Hurst exponent may vary overtime, which would result in what it is referred as multi-scaling or multi-fractal processes, in contrast to the uni-scaling or uni-fractal process used in this paper. This is not a trivial challenge since estimating the Hurst exponent requires long time-series for obtaining a feasible estimate.

Second, it is not clear how to simulate a process which captures serial-dependence and cross dependence simultaneously. When using Cholesky decomposition for transforming independent random numbers into serial-dependent random numbers, and subsequently applying the same method for transforming the latter into cross-dependent random numbers, the simulated series' Hurst exponent diverges from the assumed. An intuitive approach to circumvent this inconvenience is to estimate the Hurst exponent from portfolios' time-series and not from individual assets, and to simulate portfolios' processes directly; however, this may be impractical for some purposes.

Lastly, Cholesky method for simulating prices or returns' paths may become computationally cumbersome. In order to simulate q paths it is unavoidable to work with several matrices of size $q \times q$ (i.e. the covariance matrix and its Cholesky decomposition), which may turn computationally demanding when simulating a non-small number of periods-in-the-future (e.g more than 1,000 paths). Other approaches may serve the purpose of circumventing this shortcoming.

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